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Spatial Dependence of Correlation Functions in the Decay Problem for a Passive Scalar in a Large-Scale Velocity Field

S. S. Vergeles

Landau Institute for Theoretical Physics, Russian Academy of Sciences, Moscow, 119334 Russia
e-mail: ssver@itp.ac.ru

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Abstract—Statistical characteristics of a passive scalar advected by a turbulent velocity field are considered in the decay problem with a low scalar diffusivity κ (large Prandtl number ν/κ, where ν is kinematic viscosity). A regime in which the scalar correlation length remains smaller than the velocity correlation length is analyzed. The equal-time correlation functions of the scalar field are found to vary according to power laws and have angular singularities reflecting locally layered distribution of the scalar in space.

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1. INTRODUCTION

Statistical description of turbulent advection of a passive scalar quantity ϑ is a classical problem in turbulence theory. Examples of scalar field include deviations of tracer concentration and temperature from their mean values. A scalar is passive when the effect of the scalar field evolution on the flow is negligible. With regard to the aforementioned examples, this means that the variations of flow velocity due to tracer concentration fluctuations or thermal expansion can be ignored. This paper presents an analysis of the decay problem in which an initial scalar distribution is given and statistical characteristics of the time-varying scalar field are to be determined.

In this study, two- and three-dimensional turbulent flows are considered ($d = 2, 3$). A well-developed three-dimensional turbulent flow at a Reynolds number $Re \gg 1$ is briefly described as follows (e.g., see [1, 2]). Energy is injected into the fluid through eddies of approximate size L generated by external forcing at a rate of ϵ_v per unit mass. The fluid viscosity is negligible on length scales much larger than the viscous length

$$\eta = (\nu^3/\epsilon_v)^{1/4},$$

where ν is kinematic viscosity. In the inertial range $\eta \ll r \ll L$, kinetic energy is transferred from larger to smaller eddies via a steady cascade process. The corresponding mean velocity difference between fluid elements separated by a distance r was estimated by Kolmogorov [3] as

$$\delta v(r) \approx (\epsilon_v r)^{1/3}.$$

The cascade process transfers energy to the smallest eddies of size on the order of η. On the scale of η, viscosity is essential and kinetic energy dissipates into heat. Intermittency corrections to Kolmogorov's estimate [4] can be neglected in an approximate analysis.

Two-dimensional turbulence [5–7] is different from three-dimensional turbulence in that both kinetic energy and enstrophy are conserved in inviscid flow. The latter quantity is defined as

$$\Omega = \int d^2 r \omega^2,$$

where $\omega = \operatorname{curl} \mathbf{v}$ is vorticity. Accordingly, well-developed two-dimensional turbulence involves a downscale enstrophy cascade and an upscale kinetic-energy cascade, both starting from the forcing scale. The enstrophy cascade extends down to the viscous scale

$$\eta = \sqrt{\nu L/V} \ll L,$$

where V is the flow velocity on the forcing scale L. The energy cascade is cut off on a large scale when wall or bottom friction becomes sufficiently strong.

The present analysis is focused on the advection of a passive scalar quantity ϑ by turbulent flow on scales smaller than the viscous length η. Over such distances, the mean velocity difference is $\delta v(r) \propto r$, which corresponds to the Batchelor flow regime [8]. In the Batchelor flow, the Lagrangian trajectories of fluid particles diverge exponentially. The Lyapunov exponent λ defined as the mean logarithmic rate of divergence of Lagrangian trajectories determines the mean velocity gradient. In this study, the velocity field is assumed to be statistically isotropic in accordance with the standard treatment of turbulent flow as statistically isotropic on scales much smaller than the forcing scale even if the forcing is not isotropic.

Tracer (or heat) diffusion is assumed to be slow; i.e., the corresponding Prandtl or Schmidt number is large (Pr and Sc are defined as the ratios of ν to the tracer and heat diffusivity, respectively). Here, both diffusivities are denoted by κ; i.e., ν/κ is a large parameter. The nonuniformities of the scalar field advected by turbulent flow are smoothed out by diffusion over lengths smaller than the diffusive scale r_d. Since the Prandtl (or Schmidt) number is large, we have

$$r_d \sim \sqrt{\kappa/\lambda}.$$

Similarly, the viscous scale is

$$\eta \sim \sqrt{\nu/\lambda},$$

i.e., $r_d \ll \eta$.

The Batchelor regime with random velocity field can also be considered as a model of the so-called elastic turbulence [9]. Elastic turbulence develops in polymer solutions when the Reynolds number is low ($Re \ll 1$) and $Wi \sim 1$, where the Weissenberg number is defined as the ratio of the inverse polymer relaxation time μ to the characteristic velocity gradient:

$$\mathrm{Wi} = \mu L_0/V,$$

for a reference velocity V and a flow domain of characteristic size L_0. In elastic turbulence the dominant contribution to velocity gradient corresponds to eddies of size on the order of L_0. Therefore, the approximation $\delta v(r) \propto r$ is valid on the largest scales, and the flow Lyapunov exponent is estimated as

$$\lambda \sim V/L_0.$$

The velocity field is assumed to be statistically isotropic on the scale L_0. Since scalar diffusion in elastic turbulence is also assumed to be weak, the diffusive scale, r_d, is much smaller than L_0. If the Reynolds number is defined as

$$\mathrm{Re} = L_0 V/\nu,$$

then

$$r_d/L_0 \sim \sqrt{\kappa/\mathrm{Re}\nu}.$$

Thus, $r_d \ll L_0$ if

$$\mathrm{Pr} = \nu/\kappa \gg \mathrm{Re}^{-1}.$$

The initial scalar distribution $\vartheta_0(\mathbf{r})$ is assumed to be statistically homogeneous and isotropic, while the size l of nonuniformities of ϑ_0 is such that $r_d \ll l \ll \eta$. For elastic turbulence, it is assumed that $r_d \ll l \ll L_0$.

The significant progress made in recent theoretical studies of passive-scalar advection by random velocity fields was mainly achieved by using Kraichnan's model [10], in which the velocity field is treated as delta-correlated in time and smooth in space. Subsequently, this model was extended to fields characterized by the so-called multiscaling behavior. One of the first analyses of this kind was presented in [11], where the evolution of weak magnetic field in turbulent flow was considered. In Kraichnan's model, closed equations can be obtained for equal-time passive-scalar correlation functions. Since many correlation functions depend only on the integral characteristics of velocity field, the qualitative results obtained by using Kraichnan's model can be extended to velocity fields with finite correlation times. The current status of the theory of passive-scalar advection was reviewed in [12].

Theoretical analyses of the decay problem for Batchelor-regime turbulence were presented in [13–15]. In [13], Kraichnan's model was employed to examine the passive-scalar pair correlation function $F(r)$ in the case when there are two regions of asymptotic behavior of the velocity pair correlation function. For scales smaller than the viscous length, it was assumed that $\delta v(\mathbf{r}) \propto r$, while

$$\langle(\delta v(\mathbf{r}))^2\rangle \propto r^{2-\gamma}$$

for larger scales. According to [13], the corresponding passive-scalar correlation length is an exponentially increasing function of time:

$$r_+ \propto l e^{\lambda t}.$$

As it exceeds the viscous length, the function changes to a power law:

$$r_+ \propto t^{1/\gamma} \gg \eta.$$

Thereafter, the characteristic scale of fluctuations lies in the inertial range. These fluctuations drive a subviscous-range cascade flux down to the diffusive scale r_d. Accordingly, the spectral density varies as

$$k^{d-1}F(k) \propto 1/k$$

over the wavenumber interval

$$\eta^{-1} \ll k \ll r_d^{-1}.$$

Similar behavior of the spectral density is obtained when new passive-scalar fluctuations are permanently created on large scales [8, 10, 16]. In [14], high-order scalar correlation functions were analyzed in the limit of $\eta \longrightarrow 0$. An initial stage of decay was considered in [15], where the single-point statistical characteristics of the scalar field were studied for an arbitrary finite-time correlated velocity field.

The initial stage at which the scalar correlation length has not reached the viscous scale η, can be qualitatively characterized as follows [13, 15]. The evolution of the scalar field is described by assuming that any initial scalar distribution with a fluctuation length scale l can be represented by a superposition of blobs of size l. A blob of size l means an initial distribution ϑ_0 that does not vanish only within a region of size l. When the velocity field has a nonzero gradient, the blob expands and contracts along mutually orthogonal directions exponentially in time. Diffusion counterbalances contraction as the blob size reduces to $r \sim r_d$ in the contracting direction, whereas the blob size in the expanding direction is not affected by diffusion. After the smallest blob size has reduced to r_d, its volume increases, while the mean value of the scalar inside the blob decreases. A statistically homogeneous and isotropic initial scalar distribution can be represented as a superposition of many blobs chaotically dispersed in space. Since blobs separated by a distance much smaller than η are strained by the same velocity gradient, they are similarly oriented in space, and their shapes vary in a similar manner. Thus, the local scalar distribution becomes highly anisotropic. As the smallest blob size reduces to r_d, blobs tend to overlap, while the fluctuation intensity decreases with time elapsed. At the same time, there remain flow regions where the local tracer concentration substantially deviates from its mean value. In these regions, local velocity gradients are much weaker than the volume-averaged velocity gradient. The relative volume of these regions is small, because the probability of such velocity fluctuations is low.

In [15], single-point moments of a scalar field were calculated by finding the optimal velocity fluctuation intensity for which the scalar fluctuation intensity and the statistical weight of velocity fluctuations are equally important. The optimal fluctuation intensity turned out to be different for moments of different orders, which implies strong intermittency of the scalar field. In this study, the same initial stage of scalar decay is analyzed by a similar method, but equal-time correlation functions are considered. It is demonstrated that the pair correlation functions calculated in [13] for a delta-correlated velocity field remain qualitatively correct for a velocity field arbitrarily correlated in time. Since knowledge of the pair correlation function is not sufficient for characterizing the scalar distribution in space, higher order correlation functions are also analyzed. As a result, correlation functions are determined for an arbitrary finite-time correlated velocity field. Higher order correlation functions are shown to exhibit scaling behavior analogous to that of the pair correlation function considered in [13]. In accordance with the general tendency of the blob toward contraction and expansion, higher order correlation functions are found to have angular singularities. The existence of such singularities is independent of specific statistical characteristics of the velocity field. However, the power exponents of the angular singularities depend on the statistical characteristics of the velocity field.

2. STATEMENT OF THE PROBLEM

Evolution of a passive scalar quantity ϑ in a moving fluid is described by the advection-diffusion equation

$$\partial_t \vartheta + (\mathbf{v} \cdot \nabla)\vartheta = \kappa \Delta \vartheta. \tag{1}$$

If the scalar is a tracer concentration, then Eq. (1) corresponds to the limit case of massless tracer particles. The assumption that the scalar is passive means that the flow velocity $\mathbf{v}$ is not influenced by the dynamics of the ϑ field. Accordingly, the velocity and scalar fields are statistically independent.

The ratio of kinematic viscosity ν to scalar diffusivity κ is assumed to be large. If the scalar is a tracer concentration, then the ratio is the Prandlt number Pr. If the scalar quantity is temperature, then κ is heat diffusivity and ν/κ is the Schmidt number. In the present analysis, these parameters are equivalent. By convention,

$$\nu/\kappa = Pr \gg 1.$$

The diffusive scale r_d is defined as the length scale on which the advection and diffusion terms in Eq. (1) are comparable:

$$r_d = \sqrt{2\kappa/\lambda}. \tag{2}$$

Here, λ is the Lyapunov exponent defined as the mean logarithmic rate of divergence of two initially close Lagrangian trajectories: if $r(t)$ is the distance between Lagrangian particles, then

$$\lambda = \left\langle \frac{d\ln r}{dt} \right\rangle.$$

On scales much larger than r_d, the right-hand side of Eq. (1) can be neglected and an advection equation is obtained. On scales smaller than the diffusive scale, diffusion plays the dominant role, smoothing out nonuniformities in the scalar distribution. For a turbulent flow characterized by the viscous scale $\eta \sim \sqrt{\nu/\lambda}$, the condition $Pr \gg 1$ implies that $\eta \gg r_d$.

For elastic turbulence, $Re \ll 1$. If the characteristic size of the flow domain and the corresponding characteristic velocity are denoted by L_0 and V, respectively, then

$$Re = L_0 V/\nu.$$

Since the dominant contribution to velocity gradient corresponds to eddies of size on the order of L_0, the Lyapunov exponent is

$$\lambda \sim Re/\nu L_0^2.$$

Assuming that $r_d \ll L_0$, we have

$$Pr \gg Re^{-1}.$$

The characteristic size l of nonuniformities in the initial scalar distribution $\vartheta_0(\mathbf{r})$ is assumed to be such that $\eta \gg l \gg r_d$, which implies that the Peclet number is

$$Pe = l/r_d \gg 1.$$

The distribution of ϑ_0 is supposed to be statistically homogeneous and isotropic, with a pair correlation function

$$F_0(R) = \langle \vartheta_0(0)\vartheta_0(\mathbf{R}) \rangle$$

rapidly decreasing with increasing r for distances smaller than $r \approx l$.

The distribution ϑ_0 can be represented as a superposition of N scalar blobs of size l chaotically dispersed in space:

$$\vartheta_0(\mathbf{r}) = \sum_{i=1}^{N} a_i \Theta_0(|\mathbf{r} - \mathbf{r}_i|/l). \tag{3}$$

The initial scalar distribution in a blob is described by a spherically symmetric function $\Theta_0(|\mathbf{r}'|/l)$ decreasing for distances smaller than l. To be specific, we assume that

$$\int d\mathbf{r}' \Theta_0(r'/l) = l^d,$$

where d is the space dimension. The vectors $\mathbf{r}_i$ define randomly distributed locations of the blobs, and the random weighting factors a_i characterize scalar fluctuation amplitudes. If the flow domain is uniformly filled with blobs, then the concentration of blobs is

$$D^{-d} = N/\mathcal{V},$$

where $\mathcal{V}$ is the volume of the flow domain. The average distance between blobs is

$$D = (\mathcal{V}/N)^{1/d},$$

where $d = 2$ or 3.

Two limit cases of ϑ_0 distribution are considered here. The limit distribution denoted by ϑ_0^{g} describes strongly overlapping blobs ($D \ll l$). In this case, the scalar field can be treated as Gaussian. The volume average of the scalar is eliminated from analysis by assuming that the Corrsin invariant $\int d\mathbf{r}\, \vartheta^{\mathrm{g}}$ is zero, which implies that

$$\sum_i a_i = 0.$$

Averaging (3) over the random locations $\mathbf{r}_i$ yields the pair correlation function

$$\begin{aligned} F_0(R) &= \frac{1}{\mathcal{V}}\left[-\frac{l^{2d}}{\mathcal{V}} + \int d\mathbf{r}' \Theta(|\mathbf{R} + \mathbf{r}'|/l)\Theta(r'/l)\right] \\ &\times \sum_i a_i^2 = C_2 \int \frac{d\mathbf{r}'}{l^d} \Theta_0(|\mathbf{r}' + \mathbf{R}/2|)\Theta_0(|\mathbf{r}' - \mathbf{R}/2|), \end{aligned} \tag{4}$$

where

$$C_2 = \frac{(l/D)^d}{N} \sum_i a_i^2. \tag{5}$$

This result is obtained in the limit of $\mathcal{V} \longrightarrow \infty$. The condition $D \ll l$ ensures that the scalar field is Gaussian; in particular,

$$\frac{\langle \vartheta_0^4 \rangle}{6\langle \vartheta_0^2 \rangle^2} - 1 \propto (D/l)^3 \ll 1.$$

It is shown in Section 6 that the passive scalar correlation functions are proportional to the correlation functions of a Gaussian field in the long-time limit if $D \sim l$.

The distribution denoted by ϑ_0^{p} corresponds to the opposite limit, $D \gg l$. It is assumed that $a_i > 0$; i.e., the scalar has the same sign in all blobs. In the case of ϑ_0^{p}, statistically adequate results are obtained only when all points in the correlation function lie within the same blob, and averaging over $\mathbf{r}_i$ is required. In particular, the pair correlation function is given by expression (4), and single-point moments are calculated as follows:

$$\begin{aligned} \langle (\vartheta_0^{\mathrm{p}})^n \rangle &= C_n \int \frac{d\mathbf{r}'}{l^d} \Theta_0^n(r'/l) \\ &\approx \frac{l^d}{D^d} \left(\frac{\sum_i a_i}{N} \Theta_0(0) \right)^n, \end{aligned} \tag{6}$$

where

$$C_n = \frac{(l/D)^d}{N} \sum_i a_i^n.$$

Statistical characteristics of the scalar field advected by the flow at later times can be described in terms of the equal-time correlation functions defined as

$$\mathcal{F}_n(\{\mathbf{r}_j\}, t) = \left\langle \prod_{j=1}^{n} \vartheta(\mathbf{r}_j, t) \right\rangle, \tag{7}$$

where angle brackets denote volume averaging,

$$\mathcal{F}_n(\{\mathbf{r}_j\}, t) = \frac{1}{\mathcal{V}} \int d\mathbf{r} \prod_i \vartheta(\mathbf{r} + \mathbf{r}_i).$$

Since $\mathcal{V} \gg \eta^d$, the averaging in (7) on scales much smaller than η is equivalent to separate averaging over the initial scalar distribution and velocity-gradient statistics.

For elastic turbulence, the averaging in (7) should be interpreted somewhat differently. For example, the average in (7) corresponding to the experimental pipe flow examined in [9] is obtained by averaging over the pipe cross section combined with time averaging, the latter being equivalent to averaging over the velocity-gradient distribution for Newtonian turbulence.

3. FLUID MOTION

Consider the flow in a frame of reference moving with a fluid particle on length scales smaller than η (for Newtonian turbulence) or L_0 (for elastic turbulence). Since the Lagrangian velocity field is smooth on these scales, it can be described in terms of the velocity-gradient tensor $\hat{\sigma}$ defined by the relation

$$\mathbf{v} = \hat{\sigma}\mathbf{r}. \tag{8}$$

Assuming that the flow is incompressible, we have

$$\mathrm{tr}\hat{\sigma} = 0.$$

The random process $\hat{\sigma}$ has a finite correlation time in both Newtonian and elastic turbulence. However, the statistical characteristics of $\hat{\sigma}$ are not known. For a well-developed turbulent flow, the correlation time is comparable to the turnover time of the smallest eddies (of size on the order of η). It is assumed here that the distribution of $\hat{\sigma}$ is statistically isotropic. These assumptions are sufficient for estimating some characteristics of the flow and the advected scalar.

The Lagrangian trajectory $\mathbf{r}(t)$ of a fluid particle can be represented in terms of a linear transformation:

$$\mathbf{r}(t) = \hat{W}(t)\mathbf{r}(0), \quad \frac{d\hat{W}}{dt} = \hat{\sigma}\hat{W}. \tag{9}$$

Approximation (9) is accurate only for $r \ll \eta$ (for Newtonian turbulence) or $r \ll L_0$ (for elastic turbulence). Thus, the deformation of a fluid particle is described by the affine transformation $\hat{W}$. For incompressible flows,

$$\det\hat{W} = 1.$$

The motion of a fluid element is decomposed into expansion and rigid-body rotation by representing the linear transformation as

$$\hat{W} = \hat{N}\hat{D}\hat{\mathbb{O}}, \tag{10}$$

where $\hat{N}$ and $\hat{\mathbb{O}}$ are orthogonal matrices and $\hat{D}$ is a diagonal one.

Define

$$\rho_i = \ln D_{ii}, \quad i = 1, \dots, d.$$

The equations for ρ_i derived by combining (9) with (10) have the form

$$\dot{\rho}_i = \bar{\sigma}_{ii}, \quad \hat{\bar{\sigma}} = \hat{N}^{\mathrm{T}}\hat{\sigma}\hat{N}. \tag{11}$$

If $\det\hat{W} = 1$, then

$$\sum_i \rho_i = 0.$$

To derive equations for the orthogonal matrices in (10), define the rotation velocities

$$\Omega^{\mathrm{n}} = \hat{N}^{\mathrm{T}}\partial_t\hat{N}, \quad \Omega^{\mathrm{o}} = \partial_t\hat{\mathbb{O}}\hat{\mathbb{O}}^{\mathrm{T}}.$$

It holds that

$$\Omega^{\mathrm{n}}_{ij} = \frac{\bar{\sigma}_{ij}\exp(\rho_j - \rho_i) + \bar{\sigma}_{ji}\exp(\rho_i - \rho_j)}{2\sinh(\rho_j - \rho_i)},$$
$$\Omega^{\mathrm{o}}_{ij} = \frac{\bar{\sigma}_{ij} + \bar{\sigma}_{ji}}{2\sinh(\rho_i - \rho_j)}. \tag{12}$$

In the limit regime reached over a relatively short time $t \gtrsim \lambda^{-1}$, the fluid-element size changes differently along different eigendirections of the expansion tensor. To be specific, assume that

$$e^{\rho_1} \gg \dots \gg e^{\rho_d}.$$

In this limit, it follows from (12) that the statistical characteristics of $\hat{\bar{\sigma}}$ are independent of ρ_i. The expansion and contraction Lyapunov exponents defined as

$$\lambda_i = \langle\bar{\sigma}_{ii}\rangle \tag{13}$$

are on the order of the inverse correlation time of the process $\hat{\sigma}$. For incompressible flows,

$$\sum_i \lambda_i = 0.$$

Hereinafter, it is assumed that

$$\lambda_1 > \dots > \lambda_d.$$

The mean logarithmic rate of divergence of initially close Lagrangian trajectories defined above is $\lambda = \lambda_1$. On a time scale much larger than the correlation time of the process $\bar{\sigma}$, it follows from (11) that ρ_i is the sum of a large number (proportional to λt) of similarly distributed independent random variables. Therefore, the probability of deviation of the growth rate of ρ_i from its mean value given by (13) is described by the distribution function

$$\mathcal{P}(\rho_2, \rho_3) \approx \frac{C}{t}\exp[-tS(x_2, x_3)], \quad x_i = \frac{\rho_i}{t} - \lambda_i, \tag{14}$$

where C is a normalization constant and S is the so-called Cramer function (e.g., see [17]). The relative corrections to (14) are on the order of $1/\lambda t$. The Cramer function S is convex and has a minimum at the origin. It can be calculated exactly in Kraichnan's model with delta-correlated velocity field (see Appendix). For real turbulent flows, the function S cannot be found since the statistical characteristics of $\hat{\sigma}$ are not known. It is assumed here that the second derivative of S is on the order of λ^{-1}, because it is the only time scale of turbulent flow in the viscous range. The factor t^{-1} in (14) is of no importance for the analysis of the exponential behavior of correlation functions presented below.

In Kraichnan's model, $\lambda_2 = 0$. The value of λ_2 for well-developed turbulence governed by the Navier–Stokes equations can be found only by numerical simulation. For example, $\lambda_2 \approx 0.2\lambda_1$ was obtained in [18], where turbulence generated by large-scale random forcing was simulated under periodic boundary conditions.

In what follows, it is shown that the dependence of scalar correlation functions on the velocity field is determined by the function S, which is introduced phenomenologically. As a result, possible types of behavior of passive-scalar correlation functions depending on S are predicted. Conversely, information about S can be derived from correlation functions that are known from experiment.

The matrix $\hat{\mathbb{O}}$ defining the expansion and contraction directions evolves over a time interval $t \approx 1/\lambda$, reaching a steady-state limit when expansion becomes sufficiently large. According to (12), this limit corresponds to

$$e^{\rho_1} \gg \ldots \gg e^{\rho_d}$$

(see [19]). By virtue of flow isotropy, the matrix values are uniformly distributed on the group $O(d)$. The orthogonal matrix $\hat{N}$ substantially varies and becomes decorrelated from $\hat{\mathbb{O}}$ over a time interval on the order of λ^{-1}, and the corresponding distribution function is also uniform on the group $O(d)$.

4. AVERAGING OVER THE INITIAL SCALAR DISTRIBUTION

For an arbitrary distribution $\vartheta_0(\mathbf{r})$, a Fourier component $\tilde{\vartheta}(\mathbf{k}, t)$ of the solution to Eq. (1) combined with (8) can be written as

$$\begin{aligned}&\tilde{\vartheta}(\mathbf{k}, t)\\ &= \tilde{\vartheta}_0(\hat{W}^T\mathbf{k})\exp\left[-\frac{1}{(2\mathrm{Pe})^2}\mathbf{k}(\hat{W}\hat{\Lambda}\hat{W}^{\mathrm{T}})\mathbf{k}\right],\end{aligned} \tag{15}$$

$$\hat{\Lambda} = \int_0^t \lambda dt' \hat{W}^{-1}(t')\hat{W}^{-1,\mathrm{T}}(t'). \tag{16}$$

4.1. Evolution of an Individual Blob

For

$$\vartheta_0(\mathbf{r}) = \Theta_0(r),$$

expression (15) yields the scalar distribution

$$\begin{aligned}\Theta(\mathbf{r}, t) &= \int \frac{d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^d}\tilde{\Theta}_0(l\sqrt{\mathbf{k}\hat{W}\hat{W}^T\mathbf{k}})\\ &\times \exp\left\{-\frac{l^2}{(2\mathrm{Pe})^2}\mathbf{k}(\hat{W}\hat{\Lambda}\hat{W}^T)\mathbf{k}\right\}.\end{aligned} \tag{17}$$

Without specifying the function $\Theta_0(r)$, the blob can be described in terms of the inertia tensor $\hat{I}$, defined as

$$I^{\alpha\beta} = \frac{\int \vartheta(\mathbf{r}) r^\alpha r^\beta dr}{l^2 \int \vartheta(\mathbf{r}) dr}. \tag{18}$$

Substituting (18) into Eq. (1) and using (8), we obtain

$$\frac{d\hat{I}}{dt} = \hat{\sigma}\hat{I} + \hat{I}\hat{\sigma}^{\mathrm{T}} + \lambda/Pe^2. \tag{19}$$

Suppose that l is defined so that the initial condition is

$$\hat{I}(t=0) = \hat{1}.$$

The relation between $\hat{I}$ and $\hat{W}$ is nonlocal in time:

$$\hat{I} = \hat{W}\left[1 + \frac{\hat{\Lambda}}{\mathrm{Pe}^2}\right]\hat{W}^{\mathrm{T}}. \tag{20}$$

In view of (20), the integral over $\mathbf{k}$ in (17) has a significant value if

$$\mathbf{k}\hat{I}\mathbf{k} \lesssim 1.$$

The contribution of the region of integration where

$$\mathbf{k}\hat{I}\mathbf{k} \gg 1$$

is negligible. Thus, the distribution $\Theta(\mathbf{r}, t)$ is controlled by the tensor $\hat{I}$. We say that a vector $\mathbf{R}$ fits into a blob $\Theta(t)$ if

$$\mathbf{R}\hat{I}^{-1}\mathbf{R} \lesssim 1,$$

and does not fit into it if

$$\mathbf{R}\hat{I}^{-1}\mathbf{R} \gg 1.$$

Let us represent $\hat{I}$ as

$$\hat{I} = \hat{R}\hat{\mathcal{M}}\hat{R}^{\mathrm{T}}, \tag{21}$$

where $\hat{R}$ and $\hat{\mathcal{M}}$ are orthogonal and diagonal matrices, respectively. As time elapses, the blob transforms into a prolate ellipsoid with major axes e^{m_1}, e^{m_2}, and e^{m_3} (in decreasing order), where $e^{2m_i} = M_{ii}$. The matrix $\hat{R}$ determining the axes of the chaotically rotating ellipsoid is uniformly distributed on the group $O(3)$.

The equations for the logarithms m_i of the blob's major axes are derived from (19):

$$\begin{aligned}\dot{m}_i &= \tilde{\sigma}_{ii} + \lambda\exp(-2(p+m_i)),\\ \tilde{\sigma} &= \hat{R}^{\mathrm{T}}\hat{\sigma}\hat{R}, \quad p = \ln(\sqrt{2}Pe).\end{aligned} \tag{22}$$

The rotation velocity

$$\Omega^{\mathrm{r}} = \hat{R}^{\mathrm{T}}\partial_t\hat{R}$$

of the blob is given by the expression

$$\Omega^{r}_{ij} = \frac{\tilde{\sigma}_{ij}\exp(m_j - m_i) + \tilde{\sigma}_{ji}\exp(m_i - m_j)}{2\sinh(m_j - m_i)}. \quad (23)$$

A comparison of (23) with (12) shows that if

$$e^{m_1} \gg e^{m_2} \gg e^{m_3}$$

(the major axes differ substantially), then the statistical characteristics of $\hat{\tilde{\sigma}}$ are independent of $\hat{R}$, approaching those of $\bar{\sigma}$ by virtue of (11) in the limit of

$$e^{\rho_1} \gg e^{\rho_2} \gg e^{\rho_3}.$$

When diffusion is negligible, the scalar field is only advected by the flow and its deformation is controlled by the same $\hat{W}$, because

$$\hat{I} = \hat{W}\hat{W}^{T},$$

and

$$m_i = \rho_i.$$

Diffusion plays an important role in the dynamics of m_i when $m_i + p \lesssim 1$ because $\tilde{\sigma} \approx \lambda$. According to (14), the contraction of the blob is counterbalanced by diffusion on the diffusive time scale

$$t_d \approx \lambda^{-1}\ln \mathrm{Pe}. \quad (24)$$

At later times, the major axis continues to increase at the same rate, while the minor axis remains constant at r_d. This implies that the blob volume

$$V = \frac{l^d}{\Theta(\mathbf{r} = 0, t)}$$

exponentially increases and the mean value of the scalar inside the blob exponentially decreases accordingly. It follows from (17) that

$$V \propto l^d[\det\hat{I}]^{1/2} = l^d \exp\sum_i m_i. \quad (25)$$

4.2. Gaussian Statistics

Consider the case of a Gaussian initial field $\vartheta^{g}_{0}(\mathbf{r})$. Performing the averaging over the ensemble of ϑ^{g}_{0} in (7) for a particular realization of $\hat{\sigma}$ under initial condition (3) and using the fact that Eq. (1) is linear in ϑ, we find that the scalar field remains Gaussian and statistically uniform in space, but not statistically isotropic. The pair correlation function

$$G_2(\mathbf{R}, t) = C_2\int\frac{d\mathbf{r}'}{l^d}\Theta(\mathbf{r}' + \mathbf{R}/2, t)\Theta(\mathbf{r}' - \mathbf{R}/2, t) \quad (26)$$

is not spherically symmetric. Let us decompose correlation function (26) into a coordinate- and time-dependent parts:

$$G_2(\mathbf{R}, t) = U(\mathbf{R}, t)G_2(0, t). \quad (27)$$

Whereas $U(\mathbf{R}, t) \sim 1$ when $\mathbf{R}$ fits into the blob $\Theta(t)$, this factor is a rapidly decreasing function when $\mathbf{R}$ does not fit into the blob. Up to a factor on the order of unity that depends on details of the scalar distribution inside the blob,

$$G_2(0, t) = C_2[\det\hat{I}]^{-1/2}\Theta^2_0(0).$$

When averaged over the ensemble of ϑ^{g}_{0}, the $2n$th-order correlation functions given by (7) are factored into $(2n - 1)!!$ contributions by Wick's theorem. Each contribution corresponds to a certain combination of the set $\{\mathbf{r}_j\}$ of $2n$ points into pairs. If the points in the ith pair are separated by a distance $\mathbf{R}_i$, then the fully averaged contributions yield

$$F^{g}_{2n} = \left\langle \prod_{j=1}^{n} G_2(\mathbf{R}_i, t)\right\rangle_{\sigma}, \quad (28)$$

where the subscript σ denotes averaging over the $\hat{\sigma}$ statistics. The product of $G_2(\mathbf{R}_i, t)$ in (28) means that a substantial contribution to (28) in the correlation function $\mathcal{F}_{2n}$ defined by (7) corresponds to realizations of the process $\hat{\sigma}$ in which all vectors $\mathbf{R}_i$ fit into the blob described by (17).

4.3. Rare Fluctuations

Without loss of generality, an expression for the scalar correlation functions averaged only over the ensemble of ϑ^{p}_{0} (for widely separated blobs) is derived by assuming that one point in (7) lies at the origin:

$$\mathbf{r}_1 = 0.$$

In this case,

$$G^{p}_{n} = C_n\int\frac{d\mathbf{r}'}{l^3}\Theta(\mathbf{r}', t)\prod_{k>1}\Theta(\mathbf{r}_k + \mathbf{r}', t). \quad (29)$$

To decompose expression (29) into coordinate- and time-dependent parts, note that the correlation function vanishes if at least one of the vectors $\mathbf{r}_k$ does not fit into the blob. The time-dependent part behaves as follows:

$$\frac{1}{l^3}\int d\mathbf{r}'\Theta^n(\mathbf{r}') \propto [\det\hat{I}]^{-(n-1)/2}.$$

Comparing $\langle G^{p}_{n}\rangle_{\sigma}$ with (28), we see that contribution (28) to the correlation function $\mathcal{F}^{g}_{2n}$ defined by (7) in the problem with initial scalar distribution ϑ^{g}_{0} is proportional to the correlation function $\mathcal{F}^{p}_{n+1}$ defined by

(7) in the problem with initial scalar distribution ϑ_0^{p}. Note that the vectors $\mathbf{r}_k$ in (29) correspond to the vectors $\mathbf{R}_i$ in (28), which justifies the use of ϑ_0^{g} and the averaging of expressions (28) in the analysis presented below. Note also that the second-order correlation functions are formally similar in both limit cases of scalar distribution.

Thus, the dependence of correlation functions (7) corresponding to ϑ_0^{g} and ϑ_0^{p} on the realization of the process $\hat{\sigma}$ prior to averaging over the velocity statistics is determined by $\hat{I}$. Therefore, the averaging over the velocity statistics can be decomposed into the averaging over the orientations of the blob described by (17) (realizations of $\hat{R}$) and the averaging over the degree of expansion of the blob (realizations of the matrix $\hat{\mathcal{M}}$ defined by (21)).

5. AVERAGING OVER ROTATIONS

The averaging over the rotations $\hat{R}$ in (28) involves the factor $U(\mathbf{R})$. The averaging procedure is explained here for the three-dimensional pair correlation function. If $R \lesssim r_{\mathrm{d}}$, then $U(\mathbf{R}) \sim 1$ for any orientation in (17); therefore, the angle-averaged result is $\langle U \rangle_a \sim 1$. As the magnitude R increases, the vector $\mathbf{R}$ cannot fit into an arbitrarily oriented blob and $\langle U \rangle_{\mathrm{a}}$ decreases. When the value of R lies between the intermediate and smallest sizes of the blob, the averaging over $\hat{R}$ should be performed by taking such rotations that the vector $\mathbf{R}$ fits into a slab of thickness le^{m_3}. Thus, the averaging over $\hat{R}$ yields

$$\langle U(\mathbf{R}) \rangle_{\mathrm{a}} = \begin{cases} \dfrac{le^{m_3}}{R}, & e^{m_3} \ll \dfrac{R}{l} \ll e^{m_2}, \quad (30) \\ \dfrac{le^{m_2}}{R}\dfrac{le^{m_3}}{R}, & e^{m_2} \ll \dfrac{R}{l} \ll e^{m_1}. \quad (31) \end{cases}$$

Finally, if $R > le^{m_1}$ (R is larger than the blob size in any direction), then $\langle U \rangle_{\mathrm{a}}$ is exponentially small; i.e., all passive-scalar correlation functions vanish. The analysis presented below is developed for $R \ll le^{m_1}$. Expressions (30) and (31) can be unified into

$$\langle U(\mathbf{R}) \rangle_{\mathrm{a}} = \exp\{-\chi(-m_2 + y) - \chi(-m_3 + y)\},$$
$$\chi(x) = \begin{cases} x, & x > 0, \\ 0, & x < 0, \end{cases} \tag{32}$$

where

$$y = \ln(R/l).$$

Corrections to the argument of the exponential are of order $O(1)$. The asymptotic behavior of $\langle U \rangle_{\mathrm{a}}$ described by (32) can also be found by direct calculation using a parameterization of $\hat{R}$ in terms of angles and averaging $U(\mathbf{R})$ over the angles.

The four-point correlation function can be calculated by using a configuration of points $\mathbf{r}_i$ in (7) such that $\mathbf{R}_1 \perp \mathbf{R}_2$ in (28). Suppose that $R_1 > R_2$. In averaging the product $U(\mathbf{R}_1)U(\mathbf{R}_2)$, the ellipse with axes R_1 and R_2 must fit into the blob. If $le^{m_2} < R_2$, this cannot be done and the contribution of the corresponding realization of $\hat{I}$ vanishes. Otherwise,

$$\begin{aligned}\langle U(\mathbf{R}_1)U(\mathbf{R}_2) \rangle_{\mathrm{a}} = {} & \exp[-\chi(-m_2 + y_1) \\ & -\chi(-m_3 + y_1) - \chi(-m_3 + y_2)],\end{aligned} \tag{33}$$

where

$$y_{1,2} = \ln(R_{1,2}/l).$$

In the general case, any pair $\mathbf{R}_{1,2}$ can be used to construct linear combinations $\tilde{\mathbf{R}}_1$ and $\tilde{\mathbf{R}}_2$ such that

$$U(\tilde{\mathbf{R}}_1)U(\tilde{\mathbf{R}}_2) = U(\mathbf{R}_1)U(\mathbf{R}_2)$$

up to a factor of order unity and $\tilde{\mathbf{R}}_1 \perp \tilde{\mathbf{R}}_2$, and $R_{1,2}$ on the right-hand side of (33) should be replaced by $\tilde{R}_{1,2}$, respectively. If the ellipse defined by the mutually orthogonal vectors $\tilde{\mathbf{R}}_{1,2}$ fits into the blob, then so do the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$.

Analogously, the six-point scalar correlation function can be calculated by using three mutually orthogonal vectors in (28). To be specific, assume that

$$R_1 > R_2 > R_3.$$

The nonvanishing contributions to the six-point correlation function correspond to blobs with

$$le^{m_2} \gtrsim R_2$$

and

$$le^{m_3} \gtrsim R_3.$$

When these conditions are satisfied, the angle-averaged product $\langle \prod_i U(\mathbf{R}_i) \rangle_{\mathrm{a}}$ is independent of R_3, being formally similar to (33). When the vectors in (28) are arbitrary, three mutually orthogonal vectors $\tilde{\mathbf{R}}_i$ such that

$$\prod_i U(\tilde{\mathbf{R}}_i) = \prod_i U(\mathbf{R}_i)$$

should be constructed, and the magnitudes $\tilde{R}_i$ should be substituted for R_i in (33).

For $n \geq 4$ in (28), one can again find three mutually orthogonal vectors $\tilde{\mathbf{R}}_{1,2,3}$ such that the vectors $\mathbf{R}_i$ in (28) fit into the corresponding ellipsoid:

$$\prod_{i=1}^{n} U(\mathbf{R}_i) \approx \prod_{k=1}^{3} U(\tilde{\mathbf{R}}_k).$$

Thus, the average of a product of more than three functions U can always be reduced to the average of a product of three functions U of some mutually orthogonal vectors.

6. SINGLE-POINT SCALAR STATISTICS

If the distances between the points $\{\mathbf{r}_j\}$ in (7) are smaller than r_d, then $U(\mathbf{R}_i) \sim 1$ for each contribution to (28) irrespective of the velocity field realization. Therefore, these factors can be ignored in averaging over the velocity statistics. Single-point moments were analyzed in [15]. In this study, the averaging in single-point moments is performed again to determine the velocity fluctuations responsible for the dominant contribution to a moment. The results are then used to simplify calculations of non-single-point scalar correlation functions. According to (17),

$$\mathcal{Z}_\alpha(t) = \langle |\vartheta^\alpha(0, t)| \rangle = \beta_\alpha C_2^\alpha \langle [\det \hat{I}]^{-\alpha/4} \rangle \quad (34)$$

for an initial distribution ϑ_0^g up to a factor of order unity depending on details of distribution (17). The numerical factor

$$\beta_\alpha = \Gamma[(\alpha+1)/2] 2^{1+\alpha/2}/\sqrt{\pi}$$

is omitted in this section.

Expression (34) for $\alpha = 2$ can be interpreted as follows. The Gaussian form of the distribution ϑ_0^g implies that the inverse blob volume V^{-1} is much smaller than the concentration of blobs in expression (3). The mean value of the scalar in a particular blob is proportional to $1/V$, while its sign is either positive or negative. The number of blobs overlapping at a particular point is proportional to V. Therefore, the mean square value of the scalar is the sum of random variables,

$$\langle \vartheta^2 \rangle \propto V^{-2} V \propto [\det \hat{I}]^{-1/2}.$$

Consider a non-Gaussian initial passive-scalar distribution. Suppose that the initial concentration of blobs in (3) is relatively low: $D \sim l$. After the time interval t_d defined by (24) has elapsed, diffusion tends to smear the blobs. Therefore, at times t such that

$$\lambda(t - t_d) \gg 1,$$

the concentration of blobs approaches the limit

$$D^{-d} \gg 1/V.$$

Accordingly, the statistics of the scalar field conditioned on a particular realization of the process $\hat{\sigma}$ can be treated as Gaussian. The analysis that follows is developed for these times, because the scalar correlation functions obtained exhibit universal behavior.

To find an expression for $\mathcal{Z}_\alpha$ in terms of the function S, the averaging over the expansion statistics should be performed in (34). It is shown below that the moments $\mathcal{Z}_\alpha$ are associated with velocity fluctuations substantially different from typical realizations. Hereinafter, the process responsible for the dominant weighted contribution to a correlation function is called *optimal fluctuation.*

To find the optimal fluctuation, consider an arbitrary fluctuation under the following constraint on the process $\hat{\sigma}$: suppose that $\rho_i(t) = \lambda_i' t$, where each λ_i' is on the order of λ, but the equality $\lambda_i' = \lambda_i$ may not hold. The dominant contribution to this outcome is due to processes fluctuating about the path

$$\rho_i(t') = \lambda_i' t', \quad 0 < t' < t.$$

The fluctuation strength can be estimated as

$$\delta\rho_i(t') \approx \sqrt{\lambda t'(t - t')/t} \quad (35)$$

by using expression (14) and representing $\rho_i(t)$ as the result of consecutive transitions from zero to $\rho_i(t')$ over a time interval t' and from $\rho_i(t')$ to $\rho_i(t)$ over the interval $t - t'$.

Now, let us find the relation between $m_i(t)$ and $\rho_i(t)$ conditioned on this realization of the process $\hat{\sigma}$. First, consider the times such that $p \ll \lambda t \ll p^2$. At these times, the diffusion limit has already been reached; i.e.,

$$\exp(-\lambda_i' t) \gg Pe$$

for all $\lambda_i' < 0$. Since $\delta\rho(t') \ll p$ for typical fluctuations, the scatter of typical paths $\rho(t')$ leading to $\rho_i(t)$ is negligible as compared to p. By using (10), relation (20) is rewritten as

$$\hat{I} = \hat{N}\hat{D} \times \left\{ 1 + Pe^{-2} \hat{\mathbb{O}} \left(\int_0^t \lambda dt' [\hat{\mathbb{O}}^T \hat{D}^{-2} \hat{\mathbb{O}}](t') \right) \hat{\mathbb{O}}^T \right\} \hat{D}\hat{N}^T = \hat{R}\hat{\mathcal{M}}\hat{R}^T. \quad (36)$$

Note that the dominant contribution to the integral

$$\int_0^t \lambda dt' \exp(-2\lambda_i' t') = \frac{\lambda}{\lambda_i'}$$

corresponds to $t' \approx \lambda^{-1}$ if $\lambda_i' > 0$. If $\lambda_i' < 0$, then this integral is

$$J = (\lambda/|\lambda_i'|)\exp(2t\lambda_i') \gg Pe^2,$$

being dominated by the contribution corresponding to $t - t' \sim \lambda^{-1}$. Therefore, a significant contribution to the second term in braces in (36) is due to the decaying elements of the matrix $\hat{D}(t')$.

It is easy to see that $\hat{I}$ is determined by the values of $\hat{W}$ at times t' such that $t - t' \sim \lambda^{-1}$. To estimate the elements of $\hat{I}$, the matrix $\hat{\mathbb{O}}$ is represented as

$$\hat{\mathbb{O}}(t') \approx [1 - (t - t')\Omega^{o}(t)]\hat{\mathbb{O}}(t).$$

The elements of $\Omega^{o}(t)$ are small and can be estimated by using expression (12). First, suppose that $\lambda_2' > 0$. Then, $\hat{R}$ is equal to $\hat{N}$ up to Pe^{-2}. The larger two eigenvalues are equal ($m_{1,2} = \rho_{1,2}$), while the third eigenvalue of $\hat{I}$ is

$$m_3 = -p + \ln[|\lambda_3'|/\lambda]/2.$$

Here, the second term is a small correction neglected in the analysis presented below. In the opposite case of $\lambda_2' < 0$, $\hat{R}^{\mathrm{T}}\hat{N}$ is a random rotation about the major axis. The logarithm of the corresponding eigenvalue is $m_1 = \rho_1$, while $m_{2,3} = -p$ up to corrections of order $\ln[|\lambda_{2,3}'|/\lambda]$. Thus, to the required accuracy, it holds that

$$m_i = -p + \chi(\rho_i + p). \tag{37}$$

Now, let us use relation (37) to find $\mathfrak{T}_\alpha$. To be specific, suppose that the optimal fluctuation stretches the blob along the second eigendirection: $\lambda_2^* > 0$. Using (37), we obtain

$$\mathfrak{T}_\alpha = \langle C_2^\alpha \mathrm{Pe}^{\alpha/2}\exp\{\alpha\rho_3/2\}\rangle.$$

Finally,

$$\begin{aligned}\mathfrak{T}_\alpha &= C_2^\alpha \mathrm{Pe}^{\alpha/2}\int d\lambda_2' d\lambda_3' \exp\{-t(S - \alpha\lambda_3'/2)\} \\ &= C_2^\alpha \mathrm{Pe}^{\alpha/2}\exp(-\gamma_\alpha t),\end{aligned} \tag{38}$$

where

$$\gamma_\alpha = (S - \lambda_3'\alpha/2)_{\min}.$$

Here, S is written instead of $S(\lambda_2' - \lambda_2, \lambda_3' - \lambda_3)$ for brevity. The saddle points $\lambda_{2,3}^*(\alpha)$ of the integrand in (38) are determined by the equations

$$\partial_2 S = 0, \quad \partial_3 S = \alpha/2.$$

When $\lambda_2^* < 0$ is calculated, the starting assumption that the blob strained by optimal fluctuations expands in only one direction is violated, and the following approximation should be used:

$$\mathfrak{T}_\alpha = C_2^\alpha \mathrm{Pe}^\alpha \langle \exp[\alpha(\rho_2 + \rho_3)/2]\rangle.$$

Thus, if the equations

$$\lambda_2' = 0, \quad \partial_3 S = \alpha/2, \tag{39}$$

yield a point at which

$$\partial_2 S < 0,$$

then optimal fluctuations correspond to $\lambda_2^* > 0$, and $\mathfrak{T}_\alpha$ is given by (38). If $\partial_2 S > \alpha/2$, then optimal fluctuations correspond to $\lambda_2^* < 0$, and

$$\mathfrak{T}_\alpha = (C_2 \mathrm{Pe})^\alpha e^{-\gamma_\alpha t} \tag{40}$$

with

$$\gamma_\alpha = (S + \lambda_1'\alpha/2)_{\min}.$$

If $0 < \partial_2 S < \alpha/2$, then the largest contribution corresponds to realizations in which the diffusion limit is reached by the second Lyapunov exponent exactly at the instant t. Therefore,

$$\rho_2(t') \approx -pt'/t \tag{41}$$

at times t' such that $0 < t' < t$. As $t \longrightarrow \infty$, the asymptotic limit value $\lambda_2^* = 0$ is approached. The moment is expressed as

$$\begin{aligned}\mathfrak{T}_\alpha &= C_2^\alpha \mathrm{Pe}^{\alpha/2 + \partial_2 S}\exp(-\gamma_\alpha t), \\ \gamma_\alpha &= S - \lambda_3^*,\end{aligned} \tag{42}$$

where S and λ_3^* are taken at the point determined by (39). Note that the decay rate γ_α is a convex function of α, $\gamma_{2\alpha} < 2\gamma_\alpha$, which implies strong intermittency of the passive scalar field:

$$\langle|\vartheta|^{2\alpha}\rangle \gg \langle|\vartheta|^\alpha\rangle^2.$$

For any function S that remains finite at $\lambda_i' = 0$, there exists a critical value

$$\alpha_c = \partial_3 S(-\lambda_2, -\lambda_3)$$

such that the dependence of $\mathfrak{T}_\alpha$ on α is completely determined by C_2^α. For $\alpha > \alpha_c$,

$$\gamma_\alpha = S(-\lambda_2, -\lambda_3).$$

This means that similar optimal fluctuations correspond to all $\alpha > \alpha_c$. Since the blob size along the contracting directions reduces to r_d for the first time at the instant t, it holds that

$$\det\hat{I}(t) = 1.$$

The time dependence of $\mathfrak{T}_\alpha$ is completely determined by the statistical weight of these fluctuations.

Numerical simulations show that $\alpha_c \sim 10$ [18]. This result cannot be refined because more accurate values of the derivatives of the function S at its minimum cannot be inferred from the data presented in [18].

To obtain estimates corresponding to $t \gg p^2/\lambda$, note that if $\alpha < \alpha_c$ and the optimal value of λ_2^* is not zero, then the calculations that lead, in particular, to (38) can be repeated for $t \gg p^2/\lambda$.

Now, suppose that $\alpha < \alpha_c$ and $0 < S_2 < \alpha/2$ at the point determined by Eqs. (39). Since the statistics of the processes $\tilde{\sigma}_{ii}$ in (22) are not known in explicit form, the analysis of the logarithm m_i of the blob size is simplified by considering the following problem. The evolution of m_2 is modeled by the motion of a Brownian particle on the x axis starting from the origin $x = 0$. To estimate the contribution to $\mathcal{Z}_\alpha$ due to the realizations of $\hat{\sigma}$ in which m_2 is affected by diffusion only over a relatively short time $t' \lesssim t - \lambda^{-1}$ immediately before the instant t, an absorbing wall boundary condition is set at $x = -p$. The constraint

$$\rho_2(t)/t = -p/t \ll 1$$

on the process $\hat{\sigma}$ obviates the introduction of a drift term. The probability that the Brownian particle survives for a time $t \gg p^2/t$ decreases as $1/t$. Therefore, the fraction of realizations for which $\rho_2(t') > -p$ at all times before t among those ending up at $\rho_2(t) = -p$ is proportional to $1/t$. Accordingly, the exponential in long-time asymptotic expression (42) should be multiplied by a power of time. Thus, the moments calculated by finding optimal fluctuations are valid at all times if $n < \alpha_c$. In particular, relation (37) can be used in calculating $\mathcal{Z}_\alpha$.

Finally, consider the case of $\alpha > \alpha_c$. Since diffusion does not contribute to the optimal fluctuations corresponding to $t \ll p^2/\lambda$ (see argumentation above), we have $\rho_3(t) \approx -p$. The contribution of realizations subject to a similar constraint at $t \gg p^2/\lambda$ is estimated by analyzing the motion of a Brownian particle as a model of the evolution of m_3. Since $\rho_3 < 0$ for incompressible flows, a reflecting wall boundary condition must be set at $x = 0$ in addition to the absorbing wall at $x = -p$. The corresponding probability that the Brownian particle survives for a time $t \gg p^2/\lambda$ is proportional to $\exp[-\pi^2\lambda t/4p^2]$. Accordingly, the realizations that are affected by diffusion at all $t' < t$ will significantly contribute to $\mathcal{Z}_\alpha$ in the long-time limit. Since the blob expands and contracts to the smallest size r_d several times, moments of very high order cannot be evaluated by finding optimal fluctuations. To calculate them, the complete statistics of the process $\tilde{\sigma}$ may be required.

Henceforth, correlations (28) of order $2n$ are calculated for $2n < \alpha_c$.

7. TWO-DIMENSIONAL PROBLEM

Recalling that the distribution ϑ^g analyzed here evolves from a Gaussian initial distribution ϑ_0^g (see Section 2), we consider a two-dimension flow at $t \gg p/\lambda$].

In this case, an analysis analogous to that presented in Section 6 is much simpler to perform, because it does not involve the Lyapunov exponent associated with the intermediate eigendirection. For $\alpha < \alpha_c$,

$$\mathcal{Z}_\alpha = C_2 P e^{\alpha/2} \exp(-\gamma_\alpha t), \tag{43}$$

where

$$\gamma_\alpha = (S - \lambda_2' \alpha)_{\min}.$$

Applying an angle-averaging procedure analogous to that presented in Section 5 to the two-dimensional pair correlation function (27), we obtain

$$\langle U(\mathbf{R})\rangle_a = \exp[-\chi(-m_2 + y)], \quad y = \ln(R/l). \tag{44}$$

The four-point correlation function does not vanish when mutually orthogonal vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ lie in the same blob, in which case

$$R_2 < l e^{m_2}.$$

Therefore,

$$\langle U(\mathbf{R}_1) U(\mathbf{R}_2)\rangle_a = \exp[-\chi(-m_2 + y_1)], \quad y_i = \ln(R_i/l). \tag{45}$$

Proceeding to the calculation of correlation functions, note that the corresponding optimal fluctuations can be determined by using the optimal fluctuations found for single-point moments $\mathcal{Z}_n$. To evaluate the pair correlation function, we use relation (37) to write

$$\mathcal{F}_2(\mathbf{R}) = C_2 \int d\rho_2 \times \exp[-tS(x_2) - \chi(-m_2 + y) - (m_1 + m_2 - p)]. \tag{46}$$

Performing the integral with respect to ρ_2, we obtain the scaling law

$$\mathcal{F}_2(\mathbf{R}) = C_2 e^{-y} \int d\rho_2 \exp[-tS(x_2) + \rho_2] = \frac{l}{R}\exp(-\gamma_2 t). \tag{47}$$

Thus, the optimal fluctuations for the two-dimensional pair correlation function are similar to those for $\mathcal{Z}_2$. The factor $1/R$ in (47) results from the averaging over rotations (see Section 5), because the smallest size of the blob strained by relevant fluctuations is on the order of r_d.

Now, consider one of the six contributions to (28) for the four-point correlation function. A nonvanishing contribution is obtained only if R_2 fits into the smallest blob size $le^{m_2(t)}$ at the instant t:

$$F_{2n} = C_2^n \left\langle \frac{R_1}{le^{m_2}} \exp\left[-\frac{n(m_1+m_2)}{2}\right]\right\rangle_{le^{m_2} > R_2}, \quad (48)$$
$$n = 2.$$

To find an optimal fluctuation, consider the following class of realizations of the process $\hat{\sigma}$. Initially, the largest blob size increases:

$$m_1(t') = -\lambda_2'(t'),$$

$$0 < t' < t - \tau, \quad 0 > \lambda_2' \sim \lambda.$$

The smallest size decreases to r_d over a time interval $t_d' \approx -p/\lambda_2'$ and increases at $t_d' < t'$. Accordingly, we can write $t' > t - \tau$

$$m_2(t) = r_d e^z.$$

Assuming that $1 \ll z \ll \lambda t$, we find that the blob volume given by (25) is

$$V = \exp[-\lambda_2'(t-\tau) - p].$$

Since

$$e^{m_1} \gg e^{m_2}$$

at all $t' \gtrsim \lambda^{-1}$, the process $\tilde{\sigma}$ in (22) has statistical characteristics similar to those of $\bar{\sigma}$ in (11). To extend analysis to orders other than $n = 4$, the probability of this process is sought with a weighting factor $V^{-\alpha/2}$ ($\alpha < \alpha_c$), in accordance with (34). Using the saddle-point value $\lambda_2^*(\alpha)$ in averaging over λ_2', we obtain

$$\begin{aligned} &\mathcal{P}_\alpha(z, \tau) \\ &= \exp\{-\gamma_\alpha t - \tau[(S(z/\tau - \lambda_2) - \alpha z/2\tau) - \gamma_\alpha] \\ &\quad - \alpha z/2 + \alpha p/2\}. \end{aligned} \quad (49)$$

The optimal value of τ that maximizes $\mathcal{P}_\alpha(z, \tau)$ is found here by using a quadratic function S in order to calculate the final result:

$$S(x_2) = S_{22}x_2^2/2.$$

In this case,

$$\begin{aligned} &\mathcal{P}_\alpha(z) \propto \mathrm{Pe}^{\alpha/2}\exp(-\gamma_\alpha t) \\ &\times \exp[-z(\alpha/2 + 2|\lambda_2^*|S_{22})]. \end{aligned} \quad (50)$$

The contribution of optimal fluctuations to the correlation function is found by substituting $\mathcal{P}_4$ given by (50) into expression (48) and averaging over z:

$$\begin{aligned} &F_{2n}(\mathbf{R}_1, \mathbf{R}_2, t) \\ &= C_2^n \left\langle \frac{R_1}{r_d e^z} \mathcal{P}_{2n}(z)\right\rangle_{z \gtrsim \ln[R_2/r_d]} \quad (51) \\ &= C_2^n \mathrm{Pe}^{n-1} \\ &\times \exp(-\gamma_{2n} t)\frac{l}{R_1}\left(\frac{r_d}{R_2}\right)^{(n-1)+2|\lambda_2^*|S_{22}}, \quad (52) \end{aligned}$$

where $n = 2$. Note that the optimal value of z^* is such that

$$le^{m_2(t)} \approx R_2.$$

Expression (51) must be a good approximation of the correlation function because the contributions of different fluctuations are vanishingly small. For example, a contribution decaying as $\exp(-\gamma_{\alpha_c} t)$ is obtained when the change in the smallest size to $r_d e^z$ is modeled by using the linear function

$$m(t') = (z - p)t'/t.$$

However, no rigorous proof of (51) is available.

Contributions (28) to higher order correlation function (7) such that $6 \le 2n < \alpha_c$ are found by using the procedure for calculating F_4 in (51) for the following reason. As argued in Section 5, the averaging over rotations should be performed by changing to the product $U(\tilde{\mathbf{R}}_1)U(\tilde{\mathbf{R}}_2)$ with mutually orthogonal vectors $\tilde{\mathbf{R}}_{1,2}$. The ellipse defined by these vectors must be sufficiently large for the vectors in (28) to fit into it. Averaging over the expansion yields expressions (48)–(51) for any admissible n. Thus, contribution (28) with $n \ge 3$ is also given by (51) with $\tilde{R}_{1,2}$ instead of $R_{1,2}$.

7.1. Thin-Film Flows

As an example of two-dimensional flow, consider the flow of a film with thickness h treated as a passive scalar [20, 21]. Analogy with a standard scalar advection-diffusion problem, such as temperature decay, is incomplete because the equation for h describes hyperdiffusion rather than Fickian diffusion:

$$\partial_t h + (v\partial)h = -\tilde{\kappa}\Delta^2 h. \quad (53)$$

Changing to the Fourier representation and using the velocity field described by (8), we solve Eq. (53) for the blob evolving from an initial distribution $\Theta_0(r/l)$:

$$\begin{aligned} \tilde{\Theta}(\mathbf{k}, t) &= \tilde{\Theta}_0\left(l\sqrt{\mathbf{k}\hat{W}^{\mathrm{T}}\hat{W}^{\mathrm{T}}\mathbf{k}}\right)\exp\left\{-\frac{l^4}{\mathrm{Pe}^4}\right. \\ &\times \left.\int_0^t dt'[\mathbf{k}(\hat{W}(\hat{W}^{\mathrm{T}}\hat{W})^{-1}(t')\hat{W}^{\mathrm{T}})\mathbf{k}]^2\right\}, \end{aligned} \quad (54)$$

with the Peclet number again defined as

$$Pe = l/r_d,$$

where the diffusive scale is

$$r_d = (\tilde{\kappa}/\lambda)^{1/4}.$$

The relevant wavevectors $\mathbf{k}$ in (54) satisfy the inequality

$$\mathbf{k}\hat{l}\mathbf{k} \lesssim 1,$$

as in (17). Therefore, the product of n functions $G_2(\mathbf{R}_i, t)$ given by (26) can be averaged over the velocity ensemble by following the procedures described in Sections 5 and 7.

8. THREE-DIMENSIONAL PROBLEM

The correlation functions for a passive scalar in a three-dimensional flow are classified according to the value of $\partial_2 S$ at the point defined by (39) with an appropriate α. Recalling that a Gaussian initial distribution ϑ_0^g is considered here (see Section 2), we examine the limit of $t \gg p/\lambda$.

To evaluate the pair correlation function, we make use of relation (37) and follow the analysis of the single-point second-order moment given by (34) with $p \longrightarrow -y$. Classifying possible cases according to the value of $\partial_2 S$ at the point defined by (39) for $\alpha = 2$, we find that optimal fluctuations are independent of R if $S_2 < 0$, because the intermediate size of the blob is much larger than R. Hence, we obtain the scaling law

$$\mathcal{F}_2(\mathbf{R}, t) = C_2\exp(-\gamma_2 t)\frac{l}{R}. \tag{55}$$

Optimal fluctuations are also independent of R if $S_2 > 1$ at the point defined by (39). In this case, since the intermediate size of the blob strained by optimal fluctuations is on the order of the diffusive scale, we have

$$\mathcal{F}_2(\mathbf{R}, t) = C_2\exp(-\gamma_2 t)\frac{l^2}{R^2}. \tag{56}$$

If the optimal fluctuations corresponding to $R = 0$ are such that the diffusion limit is reached by the second Lyapunov exponent exactly at the instant t, in which case $0 < S_2 < 1$, then it follows from the condition

$$\ln(R_2/r_d) \ll \lambda t$$

that optimal fluctuations follow the path

$$\rho_2(t') \approx (t'/t)\ln(R_2/l).$$

Expanding the function S about the point defined by (39) to second order, we obtain

$$\mathcal{F}_2(R, t) = C_2\exp(-\gamma_2 t)(R/l)^{-1-S_2} \times \exp\left\{-\frac{\tilde{S}_{22}}{2t}(\ln(R/l))^2\right\}, \tag{57}$$

where

$$\tilde{S}_{22} = S_{22} - S_{23}^2/S_{33}.$$

Note that Kraichnan's model makes use of this approximation. Substituting expression (76) for S given by that model yields [13]

$$\mathcal{F}_2(R, t) = C_2\exp(-\lambda t/4)(R/l)^{-3/2}. \tag{58}$$

This result is obtained for

$$\lambda t \gg [\ln(R/l)]^2,$$

in which case the last exponential in (57) is close to unity. Note that the second-order correlation function is independent of diffusivity: expressions (55)–(57) are valid for $\kappa = 0$.

According to the analysis developed in Section 5, contribution (28) to the four-point correlation function given by (7) can be evaluated by assuming that the vectors $\mathbf{R}_{1,2}$ in (28) are such that $\mathbf{R}_1 \perp \mathbf{R}_2$. The analysis also shows that a significant contribution to the correlation function is due to realizations in which the intermediate size of the blob is not smaller than R_2. It is assumed here that $R_1 \gg R_2$. If $S_2 < 0$, then the intermediate size of the blob strained by optimal fluctuations exponentially increases with time. Therefore, the dependence on $R_{1,2}$ is entirely due to the averaging over the matrix $\hat{R}$ in (21), and

$$F_4(R_1, R_2, t) = C_2^2\exp(-\gamma_4 t)(R_1/l)^{-1}(R_2/l)^{-1}. \tag{59}$$

If $0 < S_2 < 1$, then the final value of ρ_2 is $\ln(R_1/l)$ and the result obtained in the limit of

$$\lambda t \gg [\ln(R_1/l)]^2$$

is

$$F_4(R_1, R_2, t) = C_2^2\exp(-\gamma_4 t)(R_1/l)^{-1-S_2}(R_2/l)^{-1}. \tag{60}$$

If $1 < S_2 < 2$, then ρ_2 linearly increases with time, reaching the value $\ln(R_2/l)$.

For

$$\lambda t \gg [\ln(R_2/l)]^2$$

we have

$$F_4(R_1, R_2, t) = C_2^2\exp(-\gamma_4 t)(R_1/l)^{-2}(R_2/l)^{-S_2}. \tag{61}$$

If $S_2 > 2$, then the intermediate size of the blob strained by an optimal fluctuation initially decreases to r_d over a time interval on the order of t_d and then increases to R_2 shortly before the instant t. The optimal fluctuation corresponding to the four-point correlation is found by following the analysis developed for two-dimensional flows. The only difference is that averaging should also be performed over the decreasing quantity ρ_3. Again, using

$$S = S_{ij}x_ix_j/2$$

as an example, we obtain

$$F_4(R_1, R_2, t) = C_2^2 Pe^2 \times \exp(-\gamma_4 t)(R_1/l)^{-2}(R_2/r_d)^{-1-2\tilde{S}_{22}|\lambda_2^*|}, \quad (62)$$

where

$$\tilde{S}_{22} = S_{22} - (S_{23})^2/S_{33}, \quad \lambda_2^* = \lambda_2^*(4)$$

(see Section 6). When a better fitting function S is used instead of a quadratic approximation, the power-law dependence on R_1 holds, whereas a more intricate function of R_2 is obtained.

Finally, possible types of spatial dependence of the six-point correlation function are obtained by assuming that the vectors $\mathbf{R}_i$ in (28) are mutually orthogonal and $R_1 \gg R_2 \gg R_3$. The final results are presented below for

$$S = S_{ij}x_ix_j/2.$$

By the time t, the intermediate and largest sizes of the blob strained by optimal fluctuations must be at least R_2 and R_3, respectively. If $S_2 < 0$ at the point defined by (39) for $\alpha = 6$, then we have an optimal $\lambda_2^* > 0$. The largest and intermediate sizes increase with time elapsed, while the smallest one initially decreases to r_d and then increases to R_3 over the time interval between $t - \tau$ and t. The time τ is determined by the method used in Section 7 for the four-point correlation function. If $S_2 > 3$, then both the intermediate and smallest sizes initially decrease to r_d and then increase to R_2 and R_3, respectively. The final results are

$$F_{2n} = C_2^n \mathrm{Pe}^{n-2}\exp(-\gamma_{2n}t)(R_1/l)^{-1}(R_2/l)^{-1} \times (r_d/R_3)^{(n-2)+2\tilde{S}_{33}|\lambda_3^*|}, \quad S_2 < 0, \quad (63)$$

$$F_{2n} = C_2^n \mathrm{Pe}^{2(n-1)}\exp(-\gamma_{2n}t)(R_1/l)^{-2} \times \exp\left\{-nz_2 - (n-1)z_3 - S_{ij}|\lambda_i^*|z_j - \sqrt{\frac{S_{ij}z_iz_j}{S_{ij}\lambda_i^*\lambda_j^*}}\right\}, \quad S_2 > n, \quad (64)$$

where

$$n = 3,$$

$$z_i = \ln(R_i/r_d), \quad i = 2, 3, \quad \tilde{S}_{33} = S_{33} - (S_{23})^2/S_{22}.$$

The case of $0 < S_2 < 3$ is more difficult to analyze. Consider a function S subject to the additional constraint

$$S_{23} > 0.$$

This means that the small deviations $\delta\rho_2$ and $\delta\rho_3$ associated with relevant fluctuations are anticorrelated (with weight V^{-3}). The smallest size increases from the diffusive scale at $t - \tau$ to R_3 at t. The intermediate size of the blob strained by optimal fluctuations does not reach r_d, and the corresponding ρ_2 varies linearly with time. The growth rate $\dot{\rho}_2(t')$ drops over a time interval $t \sim \lambda^{-1}$ after the instant $t - \tau$. For $0 < S_2 < 1$, the optimal value is

$$\rho_2^*(t) = \ln(R_1/l).$$

For $1 < S_2 < n$ ($n = 3$), we have

$$\rho_2^*(t) = \ln(R_2/l).$$

In the long-time limit, the final results are

$$F_{2n} = \mathrm{Pe}^{n-2}\exp(-\gamma_{2n}t)(R_1/l)^{-1-S_2}(R_2/l)^{-1} \times (r_d/R_3)^{n-2+2\tilde{S}_{33}|\lambda_3^*|}, \quad 0 < S_2 < 1, \quad (65)$$

$$F_{2n} = \mathrm{Pe}^{n-2}\exp(-\gamma_{2n}t)(R_1/l)^{-2}(R_2/l)^{-S_2} \times (r_d/R_3)^{n-2+2\tilde{S}_{33}|\lambda_3^*|}, \quad 1 < S_2 < n. \quad (66)$$

In expressions (63)–(66),

$$\lambda_i^* = \lambda_i^*(2n).$$

To determine correlation function (7) of order $2n$ such that $8 \le 2n < \alpha_c$, the required contributions (28) should be evaluated by using three mutually orthogonal vectors $\tilde{\mathbf{R}}_i$ (see Section 5), and then F_{2n} is calculated by following the procedure developed for F_6. As a result, expressions (63)–(66) with the appropriate n are obtained.

8.1. Angular Singularities

As an illustration, scalar correlation functions are now written out in explicit form.

First, consider the distribution ϑ^g for blobs separated by a distance $D \ll l$. Since ϑ_0^g is Gaussian zero-mean distribution, all odd-order correlation functions vanish (see Section 2). An expression for the fourth-order correlation is written here for a configuration of points such that

$$\mathbf{r}_{12} \parallel \mathbf{r}_{43}, \quad r_{12} \gg r_{43} \gg r_{\rm d},$$
$$r_{32} = r_{34}, \quad \mathbf{r}_{32} \perp \mathbf{r}_{43},$$

where

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j.$$

Thus, the points $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ are separated by distances on the order of r_{34}, while $\mathbf{r}_1$ is located at a much larger distance r_{12}. The correlation function is

$$\mathcal{F}_4^{\rm g}(\{\mathbf{r}_i\}, t) = C_2^n \langle [\det \hat{I}]^{-2} \{ [U(\mathbf{r}_{14})U(\mathbf{r}_{32}) + U(\mathbf{r}_{13})U(\mathbf{r}_{12})] + U(\mathbf{r}_{12})U(\mathbf{r}_{43}) \} \rangle. \tag{67}$$

The angular singularity corresponds to the limit $r_{34} \longrightarrow 0$. The first two terms in brackets in (67) are equal, while the last term is independent of r_{34}. Using expressions (59)–(61), we obtain

$$\mathcal{F}_4^{\rm g}(\{\mathbf{r}_i\}, t) = C_2^2 P e^x \exp(-\gamma_4 t) \left(\frac{r_{\rm d}}{r_{12}}\right)^a \left[2\left(\frac{r_{\rm d}}{r_{23}}\right)^b + 1\right], \tag{68}$$

where

$$1 < a < 2, \quad b = 1 \quad \text{or} \quad a = 2, \quad b > 2 \tag{69}$$

depending on the velocity statistics and

$$x = \begin{cases} a + b, & b < 2 \\ 4, & b > 2. \end{cases}$$

In the opposite limit of initial distribution ϑ^p (see Section 4), when the concentration of blobs is $c \ll 1$, each of the three contributions to (67) can be associated with a third-order correlation function. For the same configuration without the point $\mathbf{r}_4$, the correlation function

$$\mathcal{F}_3^{\rm p}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; t) = C_4 \langle [\det \hat{I}]^{-2} U(\mathbf{r}_{13}) U(\mathbf{r}_{32}) \rangle = C_4 P e^x \exp(-\gamma_4 t) \left(\frac{r_{\rm d}}{r_{12}}\right)^a \left(\frac{r_{\rm d}}{r_{23}}\right)^b \tag{70}$$

is proportional to the first term in (67).

Note that expressions (67) and (70) are valid at times when the blob size has reached the diffusive scale $r_{\rm d}$,

$$t \gg \lambda^{-1} \ln Pe.$$

9. CONCLUSIONS

Statistical characteristics of the passive scalar in the decay problem are very different from that in the case of a constantly injected scalar [8, 10, 16]. To summarize the results of this study and describe the spatial distribution of the scalar, the approach used in Section 2 is invoked here. The initial distribution $\vartheta_0(\mathbf{r})$ characterized by a correlation length l is represented by spherical blobs of diameter l chaotically dispersed in space. Recall that a spherical scalar blob means a scalar distribution $\Theta_0(r/l)$ such that $\vartheta = 0$ everywhere except for a region of diameter l. Scalar correlation functions are determined by averaging over the initial locations of the blobs in space and over the velocity gradient statistics.

The flow strains the blobs. The blobs separated by a distance smaller than the viscous scale η are similarly strained and similarly oriented in space. Since the smallest blob size cannot be smaller than the diffusive scale $r_{\rm d}$, the volume of a blob increases after its smallest size has reached the diffusive scale. The spatial distribution of the scalar is smoothed out as the scalar fluctuation intensity decreases with time elapsed.

The flow involves rare velocity fluctuations that relatively weakly strain the scalar blobs, and the scalar fluctuation amplitude remains relatively high in these blobs. Even if the initial scalar field is Gaussian, strong intermittency develops as the smallest blob size reaches the diffusive scale; in particular,

$$\langle |\vartheta^{2\alpha}| \rangle \gg \langle |\vartheta^{\alpha}|^2 \rangle.$$

Intermittency characterized by anomalous statistics of this kind is typical for turbulent systems [4] (see also [22, 23]).

In the long-time limit, scalar correlation functions strongly depend on orientation. The nature of these angular singularities can be understood by considering widely separated blobs. In this case, the dominant contributions to the equal-time scalar correlation functions are due to the regions where all points lie in the same blob. In particular, the single-point moments have the form

$$\langle (\vartheta^p)^n \rangle \approx \left(\frac{l}{D}\right)^3 [\Theta(\mathbf{r} = 0, t)]^n,$$

where $\Theta(\mathbf{r}, t)$ is the scalar distribution in a blob and D is the distance between blobs. If the points in a correlation function are separated by distances on the order of $r_{\rm d}$, then any blob averaged over its possible locations that contains at least one of the points in a correlation function contributes to the correlation function. If the distances between the points are larger than $r_{\rm d}$ and lie on the same line, then the number of contributing blobs is much smaller. Only the flow regions where blobs are aligned with this line contribute to the correlation functions. Even less numerous are the regions where scalar blobs contain all points in a correlation function if these points lie in the same plane at distances much larger than $r_{\rm d}$ from one another. Finally, there are very few blobs whose three sizes are substantially larger than the diffusive scale. In this study, the averaging over blob orientations is considered in detail. The results include the three-point scalar correlation function (Section 7.1, Eq. (70)) and general expressions corresponding to two- and three-dimensional flows (Sections 7 and 8, respectively).

The case of a zero-mean Gaussian distribution ϑ_0^{g} is also considered here. It can be represented as a superposition of chaotically dispersed spherical blobs with positive and negative scalar values. The distances between blobs are small, $D \ll l$. Since the scalar field is invariant under the change $\vartheta^{\mathrm{g}} \longrightarrow -\vartheta^{\mathrm{g}}$, only even-order scalar correlation functions do not vanish. The $2n$th-order scalar correlation function is the sum of contributions in which the $2n$ points are combined into pairs. When the averaging over blob locations is performed, significant contributions are obtained only when the points of each pair are contained in the same blob. Since the present analysis is restricted to $r \ll \eta$, all such pairs lie in blobs having similar shapes and orientations in space. This situation is somewhat analogous to the case when all points are contained in a distribution ϑ^{p}. In the case of ϑ^{p}, the correlation function of order $n + 1$ is proportional to the contribution to the $2n$th-order correlation function in the case of ϑ_0^{g} described above. Since there are $(2n)!/[2^n n!]$ combinations of $2n$ points into pairs, the spatial dependence of equal-time scalar correlation functions is more complicated in the case of ϑ^{g}. The general expression for the four-point correlation function is obtained for a special geometry (Section 8.1). Expressions for higher order correlation functions are too cumbersome to be written out here. The angular singularities arising in the case of a constantly injected scalar were studied for $r \ll l$ in [24] and for $r \gg l$ in [25] (l is the injection scale).

Note the following property of the scalar pair correlation function $\mathcal{F}(R)$: when $R \gg r_{\mathrm{d}}$, it is independent of diffusivity for arbitrary velocity statistics such that $\alpha_{\mathrm{c}} > 2$. Recall that if $\alpha > \alpha_{\mathrm{c}}$ and the initial distribution ϑ^{g} is Gaussian, then the moments $\langle|\vartheta^{\alpha}|\rangle$ are controlled by similar velocity fluctuations in which the flow is nearly frozen. The value of α_{c} depends only on the velocity statistics in the viscous range. In the case of a general velocity field, high-order correlation functions depend on diffusivity, even though the fourth-order scalar correlation function may be independent of diffusivity for some velocity statistics such that $\alpha_{\mathrm{c}} > 4$ (see expressions (59)–(61)).

Over the time interval

$$t_\eta \approx \lambda^{-1}\ln[\eta/r_{\mathrm{d}}],$$

a fluid element of initial size η contracts along a certain direction to a size on the order of r_{d}. Accordingly, approximation (9) fails for $r \gtrsim r_{\mathrm{d}}$, and the results obtained in this study are not valid. For elastic turbulence, the theory developed here fails on the time scale

$$t_L \approx \lambda^{-1}\ln[L_0/r_{\mathrm{d}}].$$

Note also that it is frequently supposed in experiments on turbulent dispersion of passive tracers [20, 21, 26] that the range of the Batchelor regime is not limited by the viscous scale and extends to the forcing scale L. It was shown in [6, 7] that velocity fluctuations in the direct-cascade range vary as

$$\delta v(r) \propto r[\ln(r/r_{\mathrm{d}})]^{1/3}.$$

The final results presented here are obtained by taking into account only the exponential in expression (14). Therefore, these results should remain valid if a logarithmic correction is introduced. However, further analysis is required to obtain a final answer to this question.

APPENDIX

In Kraichnan's model, the process $\hat{\sigma}$ in (8) is a zero-mean white noise:

$$\langle\sigma_{\alpha i}(t_1)\sigma_{\beta j}(t_2)\rangle = \mathcal{D}[(d+1)\delta_{ij}\delta_{\alpha\beta} - \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}]\delta(t_2 - t_1), \quad (71)$$

where d is the space dimension. The fluctuating part of $\hat{\tilde{\sigma}}$ in (10) is characterized by an autocorrelation function similar to (71). Since the matrix $\hat{N}(t)$ depends on $\hat{\sigma}(t)$, the mean value of $\hat{\tilde{\sigma}}$ does not vanish. To calculate $\langle\hat{\tilde{\sigma}}\rangle$, contact terms should be factored out. To do this, the rotation matrix $\hat{N}(t)$ is represented as

$$\hat{N}(t) = N(t-\delta t)\left[1 + \int_{t-\delta t}^{t} dt'\hat{\Omega}^n(t')\right].$$

Hence,

$$\langle\hat{\tilde{\sigma}}\rangle = \left\langle -\int_{t-\delta t}^{t} dt'\hat{\Omega}^n(t')\hat{N}^{\mathrm{T}}(t-\delta t)\hat{\sigma}(t)\hat{N}(t-\delta t) + \text{permuted terms}\right\rangle. \quad (72)$$

Combining (12) with the identity

$$\int^{t} dt'\delta(t'-t) = \frac{1}{2}$$

and taking the limit as $\delta t \longrightarrow 0$, we find that only diagonal elements of the matrix $\langle\hat{\tilde{\sigma}}\rangle$ do not vanish. Thus,

$$\hat{\tilde{\sigma}}_{ii} = \frac{\mathcal{D}d}{2}\sum_{j\neq i}^{d}\coth[2(\rho_i - \rho_j)] + \xi_i, \quad (73)$$

where ξ_i is Gaussian noise with autocorrelation function

$$\langle\xi_i(t)\xi_j(t')\rangle = \delta(t-t')\mathcal{D}(d\delta_{ij} - 1). \quad (74)$$

Suppose that $\rho_1 > \ldots > \rho_d$. At times $t \gg 1/\mathcal{D}$, the joint probability distribution function of ρ_i has the form

$$\mathcal{P} \propto t^{-1} e^{-tS} \delta\left(\sum_i \rho_i\right), \tag{75}$$

where S is the Cramer function

$$S = \frac{1}{2Dd}\sum_{i=1}^{d} x_i^2, \quad x_i = \frac{\rho_i}{t} - \lambda_i,$$
$$\lambda_i = \frac{d[(d+1)-2i]}{2}\mathcal{D}. \tag{76}$$

Note that if the decomposition

$$\hat{W} = \hat{N}\hat{D}\hat{T}$$

is used instead of (10), where $\hat{T}$ is an upper triangular matrix with $T_{ii} = 1$, then expression (76) is exact [27].

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