Effects of thin film and Stokes drift on the generation of vorticity by surface waves

V. M. Parfenyev,* S. S. Vergeles, and V. V. Lebedev

Landau Institute for Theoretical Physics RAS, Chernogolovka, 1-A Akademika Semenova av., 142432, Russia

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Recently a theoretical scheme explaining the vorticity generation by surface waves in liquids was developed [Phys. Rev. Lett. **116**, 054501 (2016)]. Here we study how a thin (monomolecular) film presented on the surface of liquid affects the generated vorticity. We demonstrate that the vorticity becomes parametrically larger than for the case of liquid with a free surface, and the parameter is the quality factor of surface waves up to numerical factor. We also discuss the PIV experimental scheme intended to observe the generated vorticity and find that Stokes drift influences the measured velocity field. Explicit expression for the vertical vorticity was obtained.

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I. INTRODUCTION

Boundary layers between different environments are crucial for biological, chemical, and physical processes due to extreme conditions. For example, the sea surface is believed to play an important role in the origin and earlier evolution of life on the Earth [1]. The sea-surface microlayer is known to have a very complex structure, and sometimes the most upper layer is a monomolecular surface film formed by surfactant [2]. Here we consider how such a thin film at a liquid surface modify hydrodynamic motion. It is well known that the presence of a film increases the damping of surface waves [3,4]. The history of this phenomenon dates back to antiquity, when ancient Greeks used oil to calm rough seas. The effect is related to the film incompressibility that is correct for relatively slow surface waves. The films are formed by insoluble agents and therefore the film mass is conserved. That leads to the local conservation law for the film density and, as a consequence, to the additional hydrodynamic surface squeezing mode [5,6]. The squeezing mode is faster than the gravitational-capillary waves, which justifies the incompressibility condition used at analyzing the damping of these waves.

Recently, we have analytically established a mechanism of the vertical vorticity generation by nonlinear interaction of surface waves in slightly viscous liquid with a free surface [7]. The velocity of the surface flow associated with the vertical vorticity can be estimated as $v \sim \omega k h^2$, where k is the wave vector, ω is the wave frequency, and h is the wave amplitude. Remarkably, the velocity does not depend on the viscosity of liquid, although the surface flow is produced by the viscous mechanism. The situation is different in another but similar system of freely suspended thin smectic film, performing transverse oscillations in an air environment [8]. The presence of the film changes the boundary conditions for the hydrodynamic motion. To satisfy the boundary condition posed at the film, the nonpotential contribution to the velocity field arising due to viscosity must be of the order of potential one, while the contribution is suppressed in the case of liquid with free surface. As a result, nonlinear interaction leads to the enhancement of velocity associated with the vertical vorticity up to the value $v \sim \omega \kappa h^2$, where $1/\kappa = \sqrt{\nu_a/\omega}$ is a thickness of boundary viscous sublayer and ν_a is an air kinematic viscosity. Let us stress that now the velocity depends on viscosity.

In the present paper we study the generation of vertical vorticity by surface waves in liquids, when a thin film is present at the surface—the combination of two aforementioned cases. We consider only films of negligible thickness (monomolecular), which can be formed, e.g., due to contamination of the ocean surface. The boundary conditions for the bulk flow are changed in comparison with the free surface case, and the nonpotential part of the oscillating flow is as large as in the case of freely suspended smectic films. Thus, for the surface waves of the same amplitude the generated vorticity in the contaminated case becomes parametrically larger than in the free surface case. We establish the dependence of vertical vorticity on the wave spectrum, obtain the explicit formula for it in terms of surface elevation, and make several quantitative predictions, which can be checked experimentally. All results are compared with the cases of free surface liquids and freely suspended smectic films.

The easiest way to measure the vorticity is to use the particle image velocimetry (PIV) method; see, e.g., Refs. [7,9,10]. However, the technique does not allow us to obtain surface vorticity itself. The particles floating on the surface should be treated as Lagrangian markers, which move not only horizontally, but also in a vertical direction with the surface. This vertical motion leads to the correction in measured velocity associated with the Stokes drift [11]. When analyzing the surface solenoidal currents one should take into account the effect upon treatment of the experimental data. We find the correction and show that it just changes the amplitude of the vorticity, keeping its spatial structure the same. The results are of great importance for experimentalists.

II. BASIC EQUATIONS

We consider the bulk motion of a liquid, which obeys the Navier–Stokes equation [3,4]

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\boldsymbol{\nabla} P/\rho + \nu\nabla^2\boldsymbol{v}, \tag{1}$$

where ρ and ν are the liquid mass density and the kinematic viscosity coefficient, respectively, $\boldsymbol{v}$ is the liquid velocity, and P is pressure. Equation (1) has to be supplemented by the incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$. The straightforward calculations lead to the equation for vorticity $\boldsymbol{\varpi} = \operatorname{curl} \boldsymbol{v}$,

$$\partial_t \boldsymbol{\varpi} = -(\boldsymbol{v}\nabla)\boldsymbol{\varpi} + (\boldsymbol{\varpi}\nabla)\boldsymbol{v} + \nu\nabla^2\boldsymbol{\varpi}. \tag{2}$$

One should also supplement the Navier–Stokes equation (1) by the boundary conditions posed at the liquid surface. First,

*parfenius@gmail.com

it is the kinematic boundary condition [3]

$$\partial_t h = v_z - v_x \partial_x h - v_y \partial_y h, \tag{3}$$

implying that the liquid surface moves with the velocity $\boldsymbol{v}$. Here and thereafter we assume that the axis Z is directed vertically, opposite to the gravitational acceleration $\boldsymbol{g}$ and that the equilibrium liquid surface coincides with plane $z = 0$. The deviations from the equilibrium shape are described by the elevation $h(t,x,y)$, i. e., the liquid surface is determined by the equation $z = h(t,x,y)$. Note that the pressure P in the Navier–Stokes equation (1) includes the gravitational term: $P = p + \rho g z$, where p is the internal pressure.

There is also the dynamic boundary condition that can be obtained from the requirement of zero momentum flux through the liquid surface. In the presence of a film on a liquid surface one has to take into account inhomogeneity of the surface tension coefficient $\sigma(t,x,y)$ related to its dependence on the film thickness. Therefore, the boundary conditions at a liquid surface $z = h$ are modified in comparison with the free surface; see e.g., Ref. [4]. They are

$$P - 2\rho\nu l_i l_k \partial_i v_k = \rho g h + \sigma(\nabla \boldsymbol{l}), \tag{4}$$

$$\rho\nu\delta^{\perp}_{ij} l_k(\partial_j v_k + \partial_k v_j) = \delta^{\perp}_{ij}\partial_j\sigma. \tag{5}$$

Here $\boldsymbol{l}(t,x,y) = (-\partial_x h, -\partial_y h, 1)/\sqrt{\mathrm{g}}$ is the unit vector normal to the surface, $\mathrm{g} = 1 + (\nabla h)^2$ can be thought as the determinant of the film metric tensor, and $\delta^{\perp}_{ij} = \delta_{ij} - l_i l_j$ is a projector operator on a film surface. The vorticity $\varpi_i = \epsilon_{ijk}\partial_j v_k$ should satisfy the boundary condition, which follows from Eq. (5):

$$l_m l_k \partial_k \varpi_m + (\partial_i v_k + \partial_k v_i)\epsilon_{imn} l_m K_{kn} = 0, \tag{6}$$

where ϵ_{ijk} is the unit antisymmetric tensor and we have introduced the curvature tensor $K_{ik} = K_{ki} = (\delta_{ij} - l_i l_j)\partial_j l_k$. Note that the gradient of the surface tension drops from the boundary condition (6).

To close the system of equations we need to know the dependence of surface tension σ on the film density per unit area n. Moreover, since we have a new variable n, we should write down an additional equation. This is the mass conservation law for the film density [6]:

$$\partial_t(\sqrt{\mathrm{g}}n) + \partial_\alpha(\sqrt{\mathrm{g}}n v_\alpha) = 0, \tag{7}$$

where the value of velocity field should be taken at the liquid surface $z = h$. Here and below Greek indices run over x and y. The quantity $\sqrt{\mathrm{g}}n$ is a projection of the film density on the X-Y plane. Equation (7) can be rewritten as

$$(\partial_t + \boldsymbol{v}\nabla)\ln n + \delta^{\perp}_{ij}\partial_i v_j = 0, \tag{8}$$

where the relation (3) was exploited.

III. LINEAR APPROXIMATION

Further, we consider the case where some surface waves are excited in the liquid. The case of deep water is implied. We assume that the wave steepness is small, i.e., $|\nabla h| \ll 1$. We also assume that the waves are weakly decaying, i.e., $\gamma = \sqrt{\nu k^2/\omega} \ll 1$, where ω is the wave frequency and k is its wave number.

Upon examining the hydrodynamic motion the film can be treated as incompressible in the linear approximation [4], since the surface area of the film is changed only in second order in $|\nabla h| \ll 1$. Therefore, we arrive at the surface incompressibility condition posed at $z = 0$

$$\partial_\alpha v_\alpha = 0, \quad \partial_z v_z = 0, \tag{9}$$

where we have used the three-dimensional incompressibility condition $\nabla\boldsymbol{v} = 0$. Formally, the conditions (9) can be obtained from Eq. (8) after neglecting nonlinear terms and the time derivative. Let us establish the corresponding criterion. It follows from Eq. (5) that $\delta\sigma \sim \rho\nu v$. Therefore, $\delta n \sim (\partial\sigma/\partial n)^{-1}\rho\nu v$ and we arrive at the condition

$$\omega\rho\nu \ll kn\partial\sigma/\partial n. \tag{10}$$

In the linear approximation, all quantities characterizing the surface waves can be expressed via the surface elevation h. The explicit expressions for velocity and vorticity to leading order in parameter γ are

$$v_\alpha = \nu\frac{\hat\kappa(\hat\kappa + \hat k)}{\hat k}(e^{\hat k z} - e^{\hat\kappa z})\partial_\alpha h, \tag{11}$$

$$v_z = \nu(\hat\kappa + \hat k)(\hat\kappa e^{\hat k z} - \hat k e^{\hat\kappa z})h, \tag{12}$$

$$\varpi_\alpha = \epsilon_{\alpha\beta}\frac{\hat\kappa + \hat k}{\hat k}e^{\hat\kappa z}\partial_\beta\partial_t h + O(\gamma^2), \tag{13}$$

where we introduced nonlocal operators $\hat k = (-\partial_x^2 - \partial_y^2)^{1/2}$, $\hat\kappa = (\partial_t/\nu + \hat k^2)^{1/2}$. The first terms on the right-hand sides of expressions (11) and (12) correspond to the potential part of velocity, whereas the last terms represent corrections, arising due to viscosity. The vorticity ϖ_α is located in a relatively thin layer near the surface. The depth of the layer is estimated as $\gamma/k \ll 1/k$, where $1/k$ is the penetration depth of the potential velocity.

The presence of film does not change the dispersion law of surface waves, $\omega^2 = gk + (\sigma_0/\rho)k^3$, except for the change in σ_0—an equilibrium value of surface tension σ. However, the wave damping is larger than in the clean case:

$$\frac{\mathrm{Im}\,\omega}{\omega} = \frac{\gamma}{2\sqrt{2}}. \tag{14}$$

Formally, it happens because, to satisfy the boundary condition (9), the viscous contribution to the velocity field should be of the order of potential one, see Eqs. (11) and (12), while in the free surface case the viscous contribution is smaller in parameter γ; see, e.g., Ref. [7]. Nevertheless, let us stress that the waves attenuate weakly due to the condition $\gamma \ll 1$.

Note that the result (14) can be found in Ref. [4] for capillary waves. For these waves the criterion (10) reads as $\nu\sigma^{1/2}\rho^{1/2}k^{1/2} \ll n\partial\sigma/\partial n$.

IV. NONLINEAR MECHANISM

Principally, there is a second-order contribution in ∇h to δn. Then the first term in the boundary condition (8) could be relevant for examining nonlinear effects. However, the contribution is irrelevant for the subsequent analysis, because the film density n and the surface tension σ do not enter in the boundary condition (6).

Thus, the generation mechanism of vertical vorticity is the same as in the free surface case [7], except the change in the velocity field; see Eqs. (11) and (12). To find the Z component of the vorticity, ϖ_z, we should solve the equation

$$\left(\partial_z^2 - \hat{\kappa}^2\right)\varpi_z = -\nu^{-1}\varpi_\alpha\partial_\alpha v_z, \tag{15}$$

supplemented by the boundary condition

$$\partial_z\varpi_z = \partial_\alpha h\partial_z\varpi_\alpha - \epsilon_{\alpha\gamma}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\partial_\beta\partial_\gamma h \tag{16}$$

posed at $z = 0$ and $\varpi_z \to 0$ at $z \to -\infty$. The relation (16) is just Eq. (6) written up to second order in ∇h. By using Eqs. (11)–(13), we find a solution:

$$\varpi_z(z) = \epsilon_{\alpha\beta}\left(e^{\hat{k}z}\frac{\hat{\kappa}}{\hat{k}}\partial_\beta\partial_t h\right)(e^{\hat{k}z}\partial_\alpha h) + \frac{\epsilon_{\alpha\beta}}{2}\hat{\kappa}^{-1}e^{\hat{\kappa}z}\left[(\partial_\beta\partial_t\hat{k}^{-1}h)(\hat{\kappa}\hat{k}\partial_\alpha h) - \frac{\hat{\kappa}}{\hat{k}}\partial_\beta h\partial_\alpha\partial_t\hat{k}h\right]. \tag{17}$$

Here the first term represents the tilt of the vorticity (13) due to the surface tilt and the other term is the result of rotated vorticity spreading into the bulk.

Since we consider the nonlinearity of the second order, the characteristic frequency ω_v of vorticity ϖ_z can vary from zero to the order of the surface wave frequency ω. If $\omega_v \gg \nu k^2$ then the first term in the expression (17) is leading, otherwise both terms are of the same order. Furthermore, we assume $\omega_v \ll \nu k^2$, then one can substitute $\hat{\kappa}$ by $\hat{k}$ in the prefactor before brackets in the second line of Eq. (17). So, the first contribution on the right-hand side of Eq. (17) is localized on the scale γ/k near the surface, while the second contribution penetrates deeper, on a distance $1/k$.

Let us consider the case of monochromatic pumping, when the absolute value of the wave vector is fixed. The expression for the slow vorticity ($\omega_v \ll \nu k^2$) at a liquid surface can be simplified and takes the form

$$\varpi_z(0) = \epsilon_{\alpha\beta}\left(\frac{\hat{\kappa}}{\hat{k}}\partial_\beta\partial_t h\right)\partial_\alpha h + \epsilon_{\alpha\beta}\hat{k}^{-1}(\hat{\kappa}\partial_\alpha h\partial_\beta\partial_t h). \tag{18}$$

To illustrate the relation we consider the case of two plane waves, propagating perpendicular to each other. Then the surface elevation can be modeled as

$$h = H_1\cos(\omega t - kx) + H_2\cos(\omega t - ky), \tag{19}$$

and substituting this expression into Eq. (18), we obtain

$$\varpi_z(0) = -\frac{\sqrt{2}+1}{2\gamma}H_1H_2\omega k^2\sin(kx - ky). \tag{20}$$

Note that the presented theory is correct if the higher-order nonlinear terms are small compared with the kept ones. We should estimate the nonlinear terms by using Eq. (2), where the second-order terms for the velocity, $v^{(2)}$, have to be taken into account. From Eq. (18) we find $v^{(2)} \sim \omega kh^2/\gamma$. Therefore, the nonlinear terms with $v^{(2)}$ are small if $(\boldsymbol{v}^{(2)}\nabla)\varpi_z \ll \nu\Delta\varpi_z$. Thus, in the case $\omega_v \lesssim \nu k^2$ we arrive at the condition $kh \ll \gamma^{3/2}$, which is stronger than the weak steepness condition $kh \ll 1$.

V. STOKES DRIFT

Now we analyze the motion of passive particles placed on a liquid surface and advected by the generated surface currents (18). Examining the trajectories of such particles is a natural way to observe and detect the generated vorticity. However, one should be careful upon treatment of the experimental data, because particles move not only horizontally, but also in a vertical direction with respect to the surface. And this vertical motion leads to the correction in measured velocity field associated with the Stokes mechanism [11].

The position of each particle can be characterized by a two-dimensional vector $\boldsymbol{X} = (X,Y)^T$, which obeys the equation of motion:

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{u}(t,\boldsymbol{X}), \tag{21}$$

where $\boldsymbol{u}(t,\boldsymbol{X}) \equiv \boldsymbol{v}(t,\boldsymbol{X},h(t,\boldsymbol{X}))$ is horizontal velocity at a liquid surface. Near some point $\boldsymbol{x}_0 = (x_0,y_0)^T$ we can expand the velocity field in a Taylor series:

$$\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x}_0) + \hat{G}\cdot\delta\boldsymbol{x} + \cdots. \tag{22}$$

Here $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0$ and $\hat{G}$ is a velocity gradient tensor, its four components are

$$G_{\alpha\beta} = \partial_\beta u_\alpha(t,\boldsymbol{x}_0) = \partial_\beta v_\alpha(t,\boldsymbol{x}_0,z)|_{z=h} + \partial_z v_\alpha(t,\boldsymbol{x}_0,h)\partial_\beta h. \tag{23}$$

Now we solve Eq. (21) up to second order in parameter $|\nabla h| \ll 1$ by using an iterative method. The particle displacement is $\delta\boldsymbol{X} = \delta\boldsymbol{X}_0 + \delta\boldsymbol{X}_1$, where

$$\delta\boldsymbol{X}_0 = \int\boldsymbol{u}(t,\boldsymbol{x}_0)dt, \quad \delta\boldsymbol{X}_1 = \int\hat{G}\cdot\delta\boldsymbol{X}_0 dt,$$

and we need to keep only linear terms in $\delta\boldsymbol{X}_0$ and $\hat{G}$ to calculate $\delta\boldsymbol{X}_1$.

Experimentally, the velocity field is reconstructed according to the definition $\delta\boldsymbol{X}/\delta t$. To find $\delta\boldsymbol{X}$ one should process consecutive images. If the time difference between these images is much smaller than the wave period $2\pi/\omega$, we can neglect the $\delta\boldsymbol{X}_1$ contribution and the velocity field captured by the fast camera is just $\boldsymbol{u}(t,\boldsymbol{x})$. To obtain the stationary velocity we should average the expression over time (over many pairs of images). In principle, it is also possible to use a slow camera for registration and capture only a slow motion. In this case, the time difference between consecutive images must be a multiple of the wave period $2\pi/\omega$, and we need to take $\delta\boldsymbol{X}_1$ contributions into account. For our particular case, due to the boundary condition (9), one finds $\delta\boldsymbol{X}_1 = 0$ and therefore the reconstructed velocity field will be the same.

To calculate the vertical vorticity one should take the curl from the reconstructed velocity, i.e., $\varpi_R = \langle\epsilon_{\alpha\beta}\partial_\alpha v_\beta(t,\boldsymbol{x},h(t,\boldsymbol{x}))\rangle$, where angular brackets denote averaging over time. Up to the second order, we obtain

$$v_\beta(t,\boldsymbol{x},h) = v_\beta(t,\boldsymbol{x},0) + h\partial_z v_\beta(t,\boldsymbol{x},0), \tag{24}$$

and then

$$\varpi_R = \varpi_z(0) + \langle\epsilon_{\alpha\beta}\partial_\alpha h\partial_z v_\beta(t,\boldsymbol{x},0)\rangle. \tag{25}$$

Here we drop a term $h\partial_z\varpi_z(t,\boldsymbol{x},0)$, since it is of third order in ∇h according to the boundary condition (16). The last term in expression (25) represents the correction to the previously obtained vorticity (18), which is associated with the Stokes drift. Use of Eq. (11) leads to a compact equation

$$\varpi_R = \epsilon_{\alpha\beta}\hat{k}^{-1}(\hat{\kappa}\,\partial_\alpha h\,\partial_\beta\partial_t h). \tag{26}$$

Finally, for the surface elevation given by the expression (19), we find

$$\varpi_R = -\frac{1}{2\gamma}H_1 H_2 \omega k^2 \sin(kx - ky). \tag{27}$$

VI. DISCUSSION

Comparing the results (17) and (20) with ones obtained in Ref. [7], we conclude that the vertical vorticity generation by surface waves in the presence of a thin film at a liquid surface is more effective than in the case of liquid with free surface for surface waves of the same amplitude. The vorticity amplitude is enhanced by the additional factor $\gamma^{-1} = (\omega k^2/\nu)^{1/2} \gg 1$, which is the quality factor of surface waves at the presence of the film up to a numerical factor; see expression (14). However, the quality factor is less than that for surface waves in liquid with a free surface by the same factor γ. Thus, the reverse side is that surface waves are excited less efficiently in the presence of the film. In fact these two phenomena are closely related to each other. Indeed, according to Eq. (15) the source of vertical vorticity ϖ_z is a horizontal vorticity ϖ_α slightly rotated by the velocity field. The vorticity ϖ_α is determined by the nonpotential contribution to the velocity field, and it contains an additional factor γ^{-1} due to the boundary condition (9), when a thin film is presented at a liquid surface; see Eq. (13) and Ref. [7]. Such parametric increase in the horizontal vorticity leads to more effective generation of the vertical vorticity ϖ_z and simultaneously to the stronger wave damping. We also would like to note that exactly the same enhanced mechanism of vorticity generation takes place in freely suspended thin smectic films, which perform transverse oscillations [8]. But one needs to remember that this system is very different in detail: transverse oscillations have another dispersion law and the spatial structure of eigenmodes is more complicated. So, the direct comparison with considered system is not very meaningful.

The most probable experimental scheme intended to observe the generated vertical vorticity ϖ_z is to examine trajectories of the passive particles placed on the liquid surface [7,9,10]. Then one should take into account the Stokes mechanism [11]. We showed that the Stokes drift does not misrepresent the generated vorticity. Its influence preserves the spatial structure and the sign of the vorticity, only diminishing its amplitude; compare Eqs. (18) and (26). In particular, the resulting surface motion of particles is described by expression (27) for the two perpendicular plane waves. Generally, a simple way to detect the surface flow associated with ϖ_z is to establish the direction of the motion of the particles.

It is worth here to discuss the role of Stokes drift in the case of a liquid with a free surface, where it turns out to be similar. For the specific examples of a liquid elevation $h(t,x,y)$ considered in Ref. [7], the vorticity of Lagrangian-mean flow produced by the Stokes mechanism at a liquid surface (both discussed methods of velocity measurement give the same result) has the same spatial structure as Eqs. (14) and (16) in Ref. [7], but with the prefactor $2+\sqrt{2}$ replaced by -1. Thus, the resulting prefactor in the reconstructed vorticity ϖ_R will be $1+\sqrt{2}$. Note that there is an misprint in Ref. [7]—the color legends in Figs. 2 and 4 must be inverted. Taking into account this error we obtain that the direction of motion of the particles in Fig. 4 cannot be explained by the Stokes drift alone, since this mechanism moves particles in the opposite direction. The fact can be treated as evidence of the surface flow associated with vertical vorticity.

Finally, we would like to note that capillary effects [12] can also affect the velocity field reconstructed from the motion of floating particles and they are also of great interest for experimentalists. However, the consideration of capillary effects is beyond the scope of the present paper.

VII. CONCLUSION

To summarize, we showed that the presence of a thin film on a liquid surface parametrically increases the vertical vorticity generated by surface waves in comparison with the case of a liquid with a free surface. The parameter is related to the quality factor of surface waves and an explicit expression for vorticity in terms of surface elevation (17) was obtained. The expression was analyzed for the simplest case of two perpendicular plane waves (19) and some quantitative predictions (20) were made.

Next, we analyzed the most probable PIV experimental scheme intended to observe the generated vertical vorticity and found that the Stokes mechanism should be taken into account upon treatment experimental data. It leads to the correction in measured velocity field, which was also calculated; see Eqs. (25) and (27). We believe that obtained results allow us to understand better the phenomenon of vorticity generation by surface waves and will be of great interest to experimentalists involved in this research.

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