Supplemental Material for
"Non-linear generation of vorticity by surface waves"

S.V. Filatov[1], V.M. Parfenyev[2,3], S.S. Vergeles[2,3], M.Yu. Brazhnikov[1], A.A. Levchenko[1], and V.V. Lebedev[2,3]
[1] *Institute of solid state physics RAS, Chernogolovka, 2 Academician Ossipyan str., 142432 Russia;*
[2] *Landau Institute for theoretical physics RAS, Chernogolovka, 1-A Akademika Semenova av., 142432 Russia;*
[3] *Moscow Institute of Physics and Technology, Dolgoprudny, 9 Institutskiy per., 141700 Russia.*
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DETAILS OF DERIVATION THE EXPRESSION FOR VORTICITY

As it was explained in the paper, in order to obtain the vorticity directed along $Z-$axis one should solve the equation

$$(\partial_z^2 - \hat{\kappa}^2)\varpi_z = -f, \quad f = \nu^{-1}\varpi_\alpha \partial_\alpha v_z \tag{1}$$

with the boundary conditions

$$\partial_z \varpi_z(0) = \left[\partial_\alpha h \partial_z \varpi_\alpha - \epsilon_{\alpha\gamma}(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\partial_\beta \partial_\gamma h\right]_{z=0}, \quad \varpi_z(-\infty) = 0. \tag{2}$$

Since the vorticity ϖ_α is located in a narrow layer near the water surface, then in the main order in $k/\kappa = \gamma \ll 1$ one can find:

$$f = 2\nu^{-1}\epsilon_{\alpha\beta}\partial_\alpha \partial_t h e^{\hat{\kappa}z}\partial_\beta \partial_t h, \quad \partial_z \varpi_z(0) = 2\epsilon_{\alpha\beta}\partial_\alpha h \partial_\beta \partial_t \hat{\kappa} h, \quad \varpi_z(-\infty) = 0. \tag{3}$$

The solution of the Eq. (1) is $\varpi_z = e^{\hat{\kappa}z}A(z) + e^{-\hat{\kappa}z}B(z)$, where $A' = -\hat{\kappa}^{-1}e^{-\hat{\kappa}z}(f/2)$, $B' = \hat{\kappa}^{-1}e^{\hat{\kappa}z}(f/2)$. Integrating these expressions and using relations (3) one obtains:

$$\varpi_z(z) = 2\epsilon_{\alpha\beta}(\partial_\alpha h)(e^{\hat{\kappa}z}\partial_\beta \partial_t h). \tag{4}$$

Note that the term always has small penetration depth $1/\kappa = \gamma/k \ll 1/k$. In the case of slowly varying in time vorticity ϖ_z, i.e. $\omega_\mathrm{v} \lesssim \nu k^2$, the corrections to the expressions (3) in parameter γ can produce terms, which will have penetration depths of the order of $1/k$. One should keep these terms because due to integration they can give a comparable contribution to the vorticity ϖ_z. Thus,

$$f = 2\nu^{-1}\epsilon_{\alpha\beta}(e^{\hat{k}z}\partial_\alpha \partial_t h)(e^{\hat{\kappa}z}\partial_\beta \partial_t h), \quad \partial_z \varpi_z(0) = 2\epsilon_{\alpha\beta}(\partial_\alpha h \partial_\beta \partial_t \hat{\kappa} h + \partial_\alpha \partial_\gamma h \partial_\beta \partial_\gamma \partial_t \hat{k}^{-1} h), \quad \varpi_z(-\infty) = 0, \tag{5}$$

and finally we obtain:

$$\varpi_z(z) = 2\epsilon_{\alpha\beta}(e^{\hat{k}z}\partial_\alpha h)(e^{\hat{\kappa}z}\partial_\beta \partial_t h) + 2\epsilon_{\alpha\beta}\hat{\kappa}^{-1}e^{\hat{\kappa}z}(\partial_\alpha h \partial_\beta \partial_t \hat{k} h + \partial_\alpha \partial_\gamma h \partial_\beta \partial_\gamma \partial_t \hat{k}^{-1} h). \tag{6}$$

If $\omega_\mathrm{v} \gg \nu k^2$ then the first term in the Eq. (6) is leading. Otherwise, both terms are of the same order.

INTERACTION OF WAVES HAVING NARROW SPECTRUM

In the paper we are mainly interested in the value of vorticity ϖ_z at the water surface. Taking $z = 0$ in the Eq. (6) we obtain:

$$\varpi_z(0) = 2\epsilon_{\alpha\beta}\partial_\alpha h \partial_\beta \partial_t h + 2\epsilon_{\alpha\beta}\hat{\kappa}^{-1}(\partial_\alpha h \partial_\beta \partial_t \hat{k} h + \partial_\alpha \partial_\gamma h \partial_\beta \partial_\gamma \partial_t \hat{k}^{-1} h). \tag{7}$$

Now we consider a case of two plane waves propagating in arbitrary directions. We assume that the waves have close frequencies, that is $\Delta\omega \ll \omega$, where $\Delta\omega = \omega_1 - \omega_2$, $\omega = (\omega_1 + \omega_2)/2$ and ω_1, ω_2 are frequencies of the waves. The elevation h of the water surface

$$h(\boldsymbol{r}, t) = H_1 \cos(\boldsymbol{k}_1 \boldsymbol{r} - \omega_1 t) + H_2 \cos(\boldsymbol{k}_2 \boldsymbol{r} - \omega_2 t), \tag{8}$$

where H_i is amplitude of the wave and $\boldsymbol{k}_i$ is 2d wave vector, which obeys a dispersion relation $\omega_i^2 = g|\boldsymbol{k}_i| + (\sigma/\rho)|\boldsymbol{k}_i|^3$ and $i = \overline{1, 2}$. Since the vorticity ϖ_z appears due to the nonlinearity of the second order one should expect contributions with frequencies 2ω and $\Delta\omega$ in the surface value of vorticity $\varpi_z(0)$. To calculate the fast contribution (with frequency 2ω) we should take into account only the first term in the right-hand side of Eq. (7), because the other term is smaller due to inequality

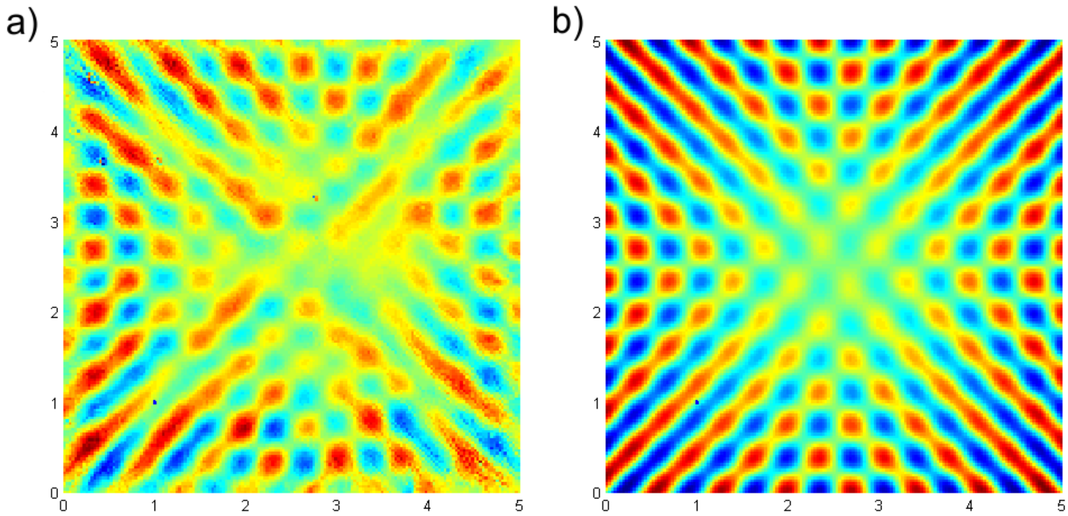

FIG. 1: a) Experimentally measured vorticity $\varpi_z(0)$ in a perfectly square cell filled by water where surface waves are excited by shaking the cell. The phase shift ψ is sufficiently smaller than unity, i.e. $\psi \ll 1$, due to the cell symmetry. b) Theoretical predictions for vorticity $\varpi_z(0)$ based on the expression (11).

$k/\kappa \sim \gamma \ll 1$. For the slow contribution (with frequency $\Delta\omega$) operator $\hat{\kappa}$ can be estimated as $\kappa \sim (\Delta\omega/\nu + k^2)^{1/2}$ and in the case $\Delta\omega \lesssim \nu k^2$ both terms in the right-hand side of Eq. (7) should be taken into account. Substituting (8) in (7) we obtain:

$$\varpi_z(0) = -H_1 H_2 |\boldsymbol{k}_1||\boldsymbol{k}_2| \sin\theta \left\{ \left[2\omega + (\omega_1|\boldsymbol{k}_1| + \omega_2|\boldsymbol{k}_2|)\hat{\kappa}^{-1}\right] \sin(\Delta\boldsymbol{k}\boldsymbol{r} - \Delta\omega t) - \Delta\omega \sin[2(\boldsymbol{k}\boldsymbol{r} - \omega t)] \right\}$$
$$- H_1 H_2 |\boldsymbol{k}_1|^2 |\boldsymbol{k}_2|^2 \sin\theta\cos\theta \left(\frac{\omega_1}{|\boldsymbol{k}_1|} + \frac{\omega_2}{|\boldsymbol{k}_2|} \right) \hat{\kappa}^{-1} \sin(\Delta\boldsymbol{k}\boldsymbol{r} - \Delta\omega t), \tag{9}$$

where $\boldsymbol{k} = (\boldsymbol{k}_1 + \boldsymbol{k}_2)/2, \Delta\boldsymbol{k} = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and θ is the angle between vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Note that the amplitude of contribution with frequency 2ω is smaller than the amplitude of contribution with frequency $\Delta\omega$, their ratio $\sim \Delta\omega/2\omega$. We also would like to pay attention that the vorticity $\varpi_z(0)$ is proportional to $\sin\theta$. Thus, the waves propagating in the same or opposite directions do not produce any vorticity $\varpi_z(0)$.

VORTICITY IN THE PERFECTLY SQUARE CELL

The Fig. 1a represents the vorticity observed in a perfectly square cell 5.0×5.0 cm^2. The figure differs significantly from the analogous figure for the near square cell which was presented and discussed in the paper. The difference is caused by the fact that in the case of perfectly square cell the phase shift ψ is sufficiently less than unity. For this reason the vorticity amplitude $\varpi_z(0)$ becomes smaller (according to the expression (14) from the paper) and more sensitive to the surface wave pattern. As we will show below, in this case the damping of surface waves plays an important role.

One can account the wave damping by modifying the dispersion relation: $\omega = -2i\nu k^2 \pm \omega_k, \omega_k^2 = gk + (\sigma/\rho)k^3$. In our experiment the frequency ω is real and it is set by the external shaker. So, to obey the dispersion relation the wave vector should be complex, we denote $\mathrm{Im}[k] = \alpha > 0$ and it means that waves attenuate as a result of propagation. In the case the elevation can be written in the form:

$$h = \frac{H_1}{2}\Big[\cos(kx - \omega t)e^{-\alpha x} + \cos(kx + \omega t)e^{\alpha x}\Big] + \frac{H_2}{2}\Big[\cos(ky - \omega t - \psi)e^{-\alpha y} + \cos(ky + \omega t + \psi)e^{\alpha y}\Big], \tag{10}$$

and substituting this expression to the Eq. (7) we obtain:

$$\varpi_z(0) = \frac{(2+\sqrt{2})}{4} H_1 H_2 \omega k^2 \Big[\sin(k(x+y)+\psi)e^{\alpha y - \alpha x} - \sin(k(x-y)+\psi)e^{-\alpha y - \alpha x} +$$
$$+ \sin(k(x-y)-\psi)e^{\alpha y + \alpha x} - \sin(k(x+y)-\psi)e^{-\alpha y + \alpha x}\Big]. \tag{11}$$

Note that the expression is distinct from zero even in the case $\psi = 0$. The result of numerical simulation based on expression (11) is presented in the Fig. 1b. Qualitatively the Figs. 1a and 1b are in a good agreement.